

ON THE FOURIER TRANSFORM OF SCHWARTZ FUNCTIONS ON RIEMANNIAN SYMMETRIC SPACES.

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ABSTRACT. Consider the (Helgason-) Fourier transform on a Riemannian symmetric space G/K . We give a simple proof of the L^p -Schwartz space isomorphism theorem ($0 < p \leq 2$) for K -finite functions. The proof is a generalization of J.-Ph. Anker's proof for K -invariant functions.

1. Introduction.

Let G/K be a Riemannian symmetric space, where G is a connected, non-compact semisimple Lie group with finite center, and K is the maximal compact subgroup fixed by a Cartan involution. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of \mathfrak{g} , the Lie algebra of G , and let \mathfrak{a} be a maximal abelian subspace of \mathfrak{p} . Let $M = Z_K(\mathfrak{a})$, then K/M is a symmetric space.

Let as usual ρ denote half the sum of the positive roots, and let W denote the Weyl group. Let $\varepsilon \geq 0$. Let \mathfrak{a}^* denote the dual of \mathfrak{a} , let $C^{\varepsilon\rho}$ be the convex hull of the set $W \cdot \varepsilon\rho$ in \mathfrak{a}^* , and let $\mathfrak{a}_\varepsilon^* = \mathfrak{a}^* + iC^{\varepsilon\rho}$ be the tube with basis $C^{\varepsilon\rho}$ in the complex dual $\mathfrak{a}_\mathbb{C}^*$.

Let \mathcal{H} denote the (Helgason-) Fourier transform on G/K , and consider the L^p -Schwartz spaces $S^p(G/K)$, $0 < p \leq 2$, and the (semi-classical) Schwartz spaces $\mathcal{S}(\mathfrak{a}_\varepsilon^* \times K/M)$. The Schwartz space isomorphism theorem ([Eg, Theorem 4.1.1]) states that:

Theorem 1. *Let $0 < p \leq 2$ and $\varepsilon = \frac{2}{p} - 1$. The Fourier transform \mathcal{H} is a topological isomorphism between $S^p(G/K)$ and $\mathcal{S}(\mathfrak{a}_\varepsilon^* \times K/M)$, with the usual inverse.*

J.-Ph. Anker gave in [An] a simple and beautiful proof of Theorem 1 when restricted to K -invariant functions, in which case the Fourier transform reduces to the spherical transform. In Section 2 of this note we extend Anker's proof to K -finite functions for the real hyperbolic spaces (the rank 1 case), and in Section 3 we sketch how to prove the general case.

Remark: This note consists of two slightly reworked Chapters from my "Progress Report": "On the Fourier transform on real Hyperbolic Spaces", Aarhus University, 1995. Recently, two articles on the same subject were published:

J. Jana and P. Sarkar, *On the Schwartz space isomorphism theorem for rank one symmetric space*, Proc. Indian Acad. Sci. Math. Sci., **117** (2007), no. 3, 333–348, and

J. Jana, *On the Schwartz space isomorphism theorem for the Riemannian symmetric spaces*, arXiv:1002.4855.

Further material may also be found in Jana's Thesis: "Isomorphism of Schwartz spaces under Fourier transform", Indian Statistical Institute, Kolkata, July 2008.

The proofs by Jana and Sarkar use a reduction to the K -invariant result by Anker; indeed they show that the various spaces of a fixed K -type are isomorphic to the similar space of trivial K -type. The proof in the present note is a more straightforward generalization of Anker's proof, with the

use of a (generalized) Abel (or Radon) transform and a cut-off function to show that the inverse transform is continuous from the Paley–Wiener space to the space of smooth functions with compact support on G/K . Unfortunately, I also conclude that the proof cannot be generalized to the general case without restriction to K -finite functions. However, I hope, and think, that the present notes could form a nice supplement to the papers and Thesis mentioned above.

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2. The rank one case.

In the following c will denote (possibly different) positive constants.

2.1. Notation and preliminaries.

Let G be a connected semisimple Lie group with finite center, and let θ be a Cartan involution of G . Then the fixed point group $K = G^\theta$ is a maximal compact subgroup. Let \mathfrak{g} and \mathfrak{k} denote the respective Lie algebras, we then have a Cartan decomposition of \mathfrak{g} given by: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The Killing form on \mathfrak{g} induces an AdK -invariant scalar product on \mathfrak{p} , and hence a G -invariant Riemannian metric on $X = G/K$. With this structure, $X = G/K$ becomes a Riemannian globally symmetric space of the noncompact type.

Fix a maximal abelian subspace \mathfrak{a} of \mathfrak{p} , and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be an Iwasawa decomposition of \mathfrak{g} . This induces the Iwasawa decomposition $G = KAN$, where A and N are the Lie groups corresponding to \mathfrak{a} and \mathfrak{n} respectively. In this Section we will assume G to be of rank one, i.e., $\dim \mathfrak{a} = 1$. In the next Section we will sketch how to remove this condition. We identify \mathfrak{a} and its dual \mathfrak{a}^* with \mathbb{R} (for the unique positive root, $\alpha \in \mathfrak{a}^*$, we make the identification $\alpha = 1$, for the unique element H in \mathfrak{a} such that $\alpha(H) = 1$, we put $H = 1$). Then we define $a_t = \exp tH$. Thus every $g \in G$ can be written as $ka_t n$, where $k \in K, t \in \mathbb{R}$ and $n \in N$ are unique. We will denote the Iwasawa projections on the K -part and A -part by $\kappa(g) = \kappa(ka_t n) = k$ and $H(g) = H(ka_t n) = t$. Furthermore we will consider the "reverse" Iwasawa decomposition, namely: $G = NAK$, where we will denote the projection onto the A -part by: $A(g) = A(na_t k) = t$. Remark that $a_{-t} = a_t^{-1}$ and $H(g) = -A(g^{-1})$.

Define $A_+ = \{a_t \in A | t > 0\}$ and $\overline{A_+} = \{a_t \in A | t \geq 0\}$, corresponding to the open and closed positive Weyl chambers, then we have the Cartan (or Polar) decomposition of G given by: $G = K\overline{A_+}K$, that is, every element $g \in G$ can be written as $k_1 a_t k_2$, where $t \in \overline{\mathbb{R}_+}$ is unique and $k_1, k_2 \in K$. We will define $|g| = |k_1 a_t k_2| = t$. For the rank one case we have the basic estimate: $|H(g)| \leq |g|$ (Given a finite-dimensional irreducible representation of G , $H(g)$ and $|g|$ can be found using a normalized highest weight vector). In the Cartan decomposition the Haar measure is given by:

$$\int_G f(x) dx = c \int_K \int_{\mathbb{R}_+} \int_K f(k_1 a_t k_2) \sinh^n t dk_1 dt dk_2,$$

where $n = \dim \mathfrak{n}$. In the hyperbolic case, $G = SO(p, 1)$, $K = SO(p)$, we get: $n = p - 1$. In $\mathfrak{a}^* \cong \mathbb{R}$ we define the element $\rho(H) = \frac{1}{2} \text{tr}(ad H|_{\mathfrak{n}})$, $H \in \mathfrak{a}$, or under the identification with \mathbb{R} : $\rho = \frac{1}{2}n$. We have the obvious estimate: $0 \leq \sinh^n t \leq ce^{2\rho t}$.

Define $M = Z_K(\mathfrak{a})$, then $B = K/M$ is a symmetric space. We will sometimes identify functions on $X = G/K$ ($B = K/M$) as right- K -invariant (right- M -invariant) functions on G (K). Then the invariant measure on X is given by:

$$\int_X f(x) dx = c \int_K \int_{\mathbb{R}_+} f(ka_t K) \sinh^n t dk dt,$$

for some constant. This is the Cartan decomposition of the measure on X . Define $A(x, b) = A(gK, kM) = A(k^{-1}g)$.

Definition 2.1.1 (The Fourier transform on G/K). For $f \in C_c^\infty(G/K)$, we define the Fourier transform by:

$$\mathcal{H}f(\nu, b) = \hat{f}(\nu, b) = \int_X f(x) e^{(-\nu+\rho)(A(x,b))} dx,$$

for all $\nu \in \mathbb{C}$, $b \in B$.

Remarks: Let $f \in C_c^\infty(K \backslash G/K)$. Then: ($k \in K$)

$$\begin{aligned} \int_X f(x) e^{(-\nu+\rho)(A(x,b))} dx &= \int_X f(k \cdot x) e^{(-\nu+\rho)(A(x,b))} dx \\ &= \int_X f(x) e^{(-\nu+\rho)(A(x,k \cdot b))} dx. \end{aligned}$$

Since K acts transitively on B , we see that \hat{f} is independent of $b \in B$. Integrating over K we get: ($b = kM$)

$$= \int_K \int_X f(x) e^{(-\nu+\rho)(A(x,kM))} dx dk = \int_X f(x) \int_K e^{(-\nu+\rho)(A(x,kM))} dk dx.$$

Let $x = gK$, we then recognize the spherical function $\varphi_{-\nu}$, $\nu \in \mathbb{C}$ on G :

$$\varphi_{-\nu}(g) = \int_K e^{(-\nu+\rho)(A(kg))} dk,$$

see [He1, Theorem 4.3]. For $f \in C_c^\infty(K \backslash G/K)$, the Fourier transform thus reduces to the spherical transform:

$$\mathcal{H}f(\nu) = \int_X f(x) \varphi_{-\nu}(x) dx.$$

treated in [An], [He1], [GV], etc.

From [He2, Chapter III, §1, §5], we get Theorems 2.1.2, 2.1.3 and 2.1.5, where $c(\cdot)$ is the Harish-Chandra c -function:

Theorem 2.1.2 (The inversion formula). Let $f \in C_c^\infty(X)$, $x \in X$. Then:

$$f(x) = c \int_{\mathbb{R}_+ \times B} e^{(i\nu+\rho)(A(x,b))} \hat{f}(i\nu, b) |c(i\nu)|^{-2} d\nu db.$$

Theorem 2.1.3 (The Plancherel formula). Let $f_1, f_2 \in C_c^\infty(X)$, then:

$$\int_X f_1(x) \overline{f_2(x)} dx = c \int_{\mathbb{R}_+ \times B} \hat{f}_1(i\nu, b) \overline{\hat{f}_2(i\nu, b)} |c(i\nu)|^{-2} d\nu db.$$

The Fourier transform extends to an isometry of $L^2(X)$ onto $L^2(i\mathbb{R}_+ \times B, c|c(i\nu)|^{-2} d\nu db)$.

A C^∞ -function $\psi(z, b)$ on $\mathbb{C} \times B$, holomorphic in z , is called a holomorphic function of uniform exponential type R , if there exists a constant $R \geq 0$ such that for each $N \in \mathbb{N}$ we have:

$$\sup_{z \in \mathbb{C}, b \in B} e^{-R|\operatorname{Re} z|} (1 + |z|)^N |\psi(z, b)| < \infty.$$

Definition 2.1.4. The space of holomorphic functions of uniform exponential type R will be denoted $\mathcal{H}^R(\mathbb{C} \times B)$. Furthermore denote by $\mathcal{H}(\mathbb{C} \times B)$ their union over all $R > 0$. Let $\mathcal{H}_e(\mathbb{C} \times B)$ denote the space of functions $\psi \in \mathcal{H}(\mathbb{C} \times B)$ satisfying the symmetry condition (SC1):

$$\int_B e^{(-\nu+\rho)(A(x,b))} \psi(-\nu, b) db = \int_B e^{(\nu+\rho)(A(x,b))} \psi(\nu, b) db, \quad \nu \in \mathbb{C}, x \in X.$$

We usually call $\mathcal{H}(\mathbb{C} \times B)$ the Paley-Wiener space. For $\psi \in \mathcal{H}_e(\mathbb{C} \times B)$ independent of b , (SC1) reduces to the symmetry condition: $\varphi_{-\nu}(x)\psi(-\nu) = \varphi_\nu(x)\psi(\nu)$, where φ_ν is the spherical function indexed by ν . Since $\varphi_{-\nu} = \varphi_\nu$ this again reduces to ψ being an even function of ν , that is, the usual symmetry condition for the spherical transform (Weyl group invariance), see [He1, p. 450].

Consider $f \in C_c^\infty(X)$. By the Cartan (polar) decomposition, the polar distance from the point $x = gK = ka_t$ to the origin $x_o = eK$ is $|x| = |t|$. Let $R > 0$. We say that $\text{supp}(f) \subset \bar{B}(0, R)$ if and only if the function f has support inside $K \times \bar{B}(0, R)$ or if and only if $f(x) = 0$ for $|x| > R$.

The Paley-Wiener Theorem 2.1.5. The Fourier transform is a bijection of the space $C_c^\infty(X)$ onto the space $\mathcal{H}_e(\mathbb{C} \times B)$, the inverse transform being given by Theorem 2.1.2. Moreover $\text{supp}(f) \subset \bar{B}(0, R)$ if and only if $\hat{f} \in \mathcal{H}_e^R(\mathbb{C} \times B)$.

We decompose the Fourier transform into two transforms. Let $f \in C_c^\infty(G/K)$, then the Radon transform is defined as follows:

$$\mathcal{R}f(t, k) = e^{\rho t} \int_N f(ka_t n) dn.$$

Proposition 2.1.6. The Radon transform \mathcal{R} maps $C_c^\infty(G/K)$ into $C_c^\infty(\mathbb{R} \times B)$. If $\text{supp} f \subset \bar{B}(0, R)$, then $\text{supp} \mathcal{R}f \subset \bar{B}(0, R) \times B$.

Proof. Let $k, k_1, k_2 \in K, n \in N, t \in \mathbb{R}$, we then get: $|ka_t n| \geq |t| = |k_1 a_{|t|} k_2|$, hence $\text{supp} \mathcal{R}f \subset \text{supp} f \times B$. \square

Remark: For $f \in C_c^\infty(K \backslash G/K)$, the Radon transform reduces to the Abel transform, see [An].

Let $\phi \in C_c^\infty(\mathbb{R} \times B)$, then the "classical" Fourier transform on $\mathbb{R} \times B$ is defined as:

$$\mathcal{F}\phi(\nu, b) = \int_{\mathbb{R}} \phi(t, b) e^{-\nu t} dt, \quad \nu \in i\mathbb{R}, b \in B.$$

Let ψ be a nice function on $i\mathbb{R} \times B$, then the "classical" inverse Fourier transform is defined by:

$$\mathcal{F}^{-1}\psi(t, b) = \frac{1}{2\pi} \int_{\mathbb{R}} \psi(i\nu, b) e^{i\nu t} d\nu, \quad t \in \mathbb{R}, b \in B.$$

We then have: ($b = kM$)

$$\hat{f}(\nu, kM) = \int_{\mathbb{R}} e^{-\nu t} \left\{ e^{\rho t} \int_N f(ka_t n) dn \right\} dt,$$

i.e., we have the following commutative diagram:

$$\begin{array}{ccc} & \mathcal{H}_e(\mathbb{C} \times B) & \\ \mathcal{H} \nearrow & & \nwarrow \mathcal{F} \\ C_c^\infty(G/K) & \xrightarrow{\mathcal{R}} & \mathcal{R}C_c^\infty(G/K) \subset C_c^\infty(\mathbb{R} \times B) \end{array}$$

The commutativity is an easy consequence of the definitions of the various transforms. Furthermore we get:

Proposition 2.1.7. *The Radon transform \mathcal{R} is an isomorphism between $C_c^\infty(G/K)$ and $\mathcal{R}C_c^\infty(G/K) = \mathcal{F}^{-1}\mathcal{H}_e(\mathbb{C} \times B) \subset C_c^\infty(\mathbb{R} \times B)$. Moreover $\text{supp} f \subset \bar{B}(0, R)$ if and only if $\text{supp} \mathcal{R}f \subset \bar{B}(0, R) \times B$.*

Proof. The Paley-Wiener Theorems above and below. \square

Remark: We have indirectly introduced a symmetry condition for functions in $C_c^\infty(\mathbb{R} \times B)$, namely that the Fourier transformed function should be in $\mathcal{H}_e(\mathbb{C} \times B)$. For functions of a specific K -type (K acting on the B -variable), this symmetry condition becomes somewhat easier to describe, see later. For the trivial K -type, or for B -invariant functions, it reduces to: $g(-t) = g(t)$, $g \in C_c^\infty(\mathbb{R})$, or g even, that is, as in [An], where the Abel transform maps $C_c^\infty(K \backslash G/K)$ onto even functions in $C_c^\infty(\mathbb{R})$.

Theorem 2.1.8 (A Paley-Wiener Theorem on $\mathbb{R} \times B$). *Let $\phi \in C_c^\infty(\mathbb{R} \times B)$ have support in $\bar{B}(0, R) \times B$, and let $f(\nu, b) = \int_{\mathbb{R}} \phi(t, b) e^{-\nu t} dt$, $\nu \in i\mathbb{R}$, $b \in B$. Then $f \in C^\infty(\mathbb{C} \times B)$, and $f(\cdot, b)$ is an entire function for fixed b , such that for all $N \in \mathbb{N}$, we have:*

$$\sup_{z \in \mathbb{C}, b \in B} e^{-R|\text{Re} z|} (1 + |z|)^N |f(z, b)| < \infty.$$

Conversely, let $f \in C^\infty(\mathbb{C} \times B)$ satisfy the above. Then there exists a function $\phi \in C_c^\infty(\mathbb{R} \times B)$, with support in $\bar{B}(0, R) \times B$, such that $f = \mathcal{F}\phi$.

Proof. An easy generalization of [Ru, Theorem 7.22]. \square

We will in the following sections need some estimates on the spherical functions φ_ν introduced before and on the Harish-Chandra c -function.

Lemma 2.1.9. *The spherical functions are all bi- K -invariant, $\varphi_{-\nu} = \varphi_\nu$, and:*

- i) *For $t \geq 0$, $\exists c > 0$ such that: $e^{-\rho t} < \varphi_o(a_t) < c(1+t)e^{-\rho t}$.*
- ii) *Let $\varepsilon \geq 0$, $|\text{Re} \nu| \leq \varepsilon \rho$ and $t \geq 0$. Then: $|\varphi_\nu(a_t)| \leq c(1+t)e^{(\varepsilon-1)\rho t}$.*

For the c -function we have:

- iii) *Let $\text{Re} \nu \geq 0$. $\exists \gamma \geq 0$ such that: $|c(\nu)|^{-1} \leq \gamma(1+|\nu|)^{\frac{n}{2}}$.*

Proof. i) [He1, Chap IV, Ex.B1; GV, Sect.4.6].

ii) Assume $\nu \geq 0$: Then:

$$\varphi_\nu(a_t) \leq e^{\nu t} \varphi_o(a_t) \leq c e^{\varepsilon \rho t} (1+t) e^{-\rho t} = c(1+t) e^{(\varepsilon-1)\rho t}.$$

iii) Properties of the Γ -function, see [He1, Chapter IV, Proposition 7.2]. \square

2.2. Schwartz spaces and dense subspaces.

We have come to the definition of the Schwartz spaces. Let $U(\mathfrak{g})$ denote the universal enveloping algebra of \mathfrak{g} . The elements of $U(\mathfrak{g})$ act on $C^\infty(G)$ as differential operators on both sides. We shall write $f(D; g; E)$ for the action of $(D, E) \in U(\mathfrak{g}) \times U(\mathfrak{g})$ on $f \in C^\infty(G)$ at $g \in G$, more explicitly we have:

$$f(D; g; E) = \left(\frac{\partial}{\partial s_1} \cdots \frac{\partial}{\partial s_d} \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_e} \right) \Big|_{s_1=\dots=s_d=t_1=\dots=t_e=0} \times f((\exp s_1 X_1) \cdots (\exp s_d X_d) g (\exp t_1 Y_1) \cdots (\exp t_e Y_e)).$$

if $D = X_1 \cdots X_d$, $E = Y_1 \cdots Y_e$ ($X_1, \dots, X_d, Y_1, \dots, Y_e \in \mathfrak{g}$).

Definition 2.2.1. Let $0 < p \leq 2$. The L^p -Schwartz space $\mathcal{S}^p(G/K)$ is the space of all functions $f \in C^\infty(G/K)$, such that

$$\sup_{g \in G} (1 + |g|)^N \varphi_o(g)^{-\frac{2}{p}} |f(D; g; E)| < \infty,$$

for any $D, E \in U(\mathfrak{g})$, and any nonnegative integer N . Here φ_o is the spherical function with $\nu = 0$.

The topology of $\mathcal{S}^p(G/K)$ is defined by the seminorms:

$$\sigma_{D,E,N}^p(f) = \sup_{g \in G} (1 + |g|)^N \varphi_o(g)^{-\frac{2}{p}} |f(D; g; E)|.$$

Remarks 1) Consider the Cartan decomposition of the integral of X , then:

$$\int_X f(x) dx = \int_{K \times \mathbb{R}_+} f(ka_t) \sinh^n t dk dt.$$

From above we see that the factor $\varphi_o(g)^{-\frac{2}{p}}$ will control the factor $\sinh^n t$, and hence the definition resembles the natural definition of a Schwartz space on $K \times \mathbb{R}_+$.

2) In fact we see that: $\mathcal{S}^p(G/K) \subset L^q(G/K)$ for $0 < p \leq q \leq 2$, while $\mathcal{S}^p(G/K) \not\subset L^q(G/K)$ for $0 < q < p \leq 2$. For $0 < p \leq q \leq 2$ the inclusion above is continuous.

3) Obviously $C_c^\infty(G/K) \subset \mathcal{S}^p(G/K)$ for $0 < p \leq 2$, and the inclusion is continuous.

Lemma 2.2.2. Let $0 < p \leq q \leq 2$. Then:

- (i) $\mathcal{S}^p(G/K)$ is a Fréchet space.
- (ii) $C_c^\infty(G/K)$ is a dense subspace of $\mathcal{S}^p(G/K)$.
- (iii) $\mathcal{S}^p(G/K)$ is a dense subspace of $\mathcal{S}^q(G/K)$.
- (iv) $\mathcal{S}^p(G/K)$ is a dense subspace of $L^q(G/K)$.
- (v) $\mathcal{S}^p(K \backslash G/K)$ is a closed subspace of $\mathcal{S}^p(G/K)$ in the Fréchet topology.

Proof. See [GV, sect. 6.1, 7.8]. □

For $E \in U(\mathfrak{k})$ define: $Ef(k) = f(k; E)$. Let Ω_K denote the Casimir element of $U(\mathfrak{k})$, then for the Laplace-Beltrami operator Δ_B on B we have (modulo a constant): $\Delta_B f = f(\cdot; \Omega_K) = f(\Omega_K; \cdot)$ ($\Omega_K \in \mathfrak{Z}(\mathfrak{k})$). Furthermore $(P(\frac{\partial}{\partial \nu}), E)$ will denote differentiation with $P(\frac{\partial}{\partial \nu})$ on the ν -variable and with E on the k -variable.

Definition 2.2.3. Fix $\varepsilon \geq 0$. Let $i\mathbb{R}_\varepsilon = i\mathbb{R} + [-\varepsilon\rho, \varepsilon\rho]$. The Schwartz space $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)$ consists of all complex valued functions $f \in C^\infty(i\mathbb{R}_\varepsilon \times B)$ such that:

- (i) For fixed $b \in B$, $f(\cdot, b)$ is holomorphic in the interior of $i\mathbb{R}_\varepsilon$ ($i\mathbb{R}_\varepsilon^\circ$).
- (ii) f and all its derivatives extend continuously to $i\mathbb{R}_\varepsilon \times B$.
- (iii) For any polynomial P , any $E \in U(\mathfrak{k})$ and nonnegative integer N we have the estimate:

$$\sup_{\nu \in i\mathbb{R}_\varepsilon, k \in K} (1 + |\nu|)^N \left| \left(P\left(\frac{\partial}{\partial \nu}\right), E \right) f(\nu, k) \right| < \infty.$$

The topology of $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)$ is defined by the seminorms:

$$\tau_{P,E,N}^\varepsilon(f) = \sup_{\nu \in i\mathbb{R}_\varepsilon, k \in K} (1 + |\nu|)^N \left| \left(P\left(\frac{\partial}{\partial \nu}\right), E \right) f(\nu, k) \right| < \infty.$$

Remarks: For $\varepsilon = 0$ condition i) is empty. For $\varepsilon > 0$ ii) and iii) are equivalent to:

(iv) For any polynomial P , any $E \in U(\mathfrak{k})$ and any nonnegative integer N we have the estimate:

$$\sup_{\nu \in i\mathbb{R}_\varepsilon^o, k \in K} (1 + |\nu|)^N \left| \left(P \left(\frac{\partial}{\partial \nu} \right), E \right) f(\nu, k) \right| < \infty.$$

To see this, observe that for fixed $k \in K$, $(P(\frac{\partial}{\partial \nu}), E) f(\nu, k)$ is bounded on $i\mathbb{R}_\varepsilon^o$ for all polynomials P . $i\mathbb{R}_\varepsilon^o$ is convex, hence by the Mean Value Theorem we get:

$$|f(x_1, k) - f(x_2, k)| \leq \left\{ \sup_{\nu \in i\mathbb{R}_\varepsilon^o} |\nabla f(\nu, k)| \right\} |x_1 - x_2|, \quad x_1, x_2 \in i\mathbb{R}_\varepsilon^o,$$

where ∇ is the operator $\nabla = \frac{df}{d\nu}$, which shows that $f(\cdot, k)$ is uniformly continuous on $i\mathbb{R}_\varepsilon^o$, hence by density $f(\cdot, k)$ extends to a uniformly continuous function on $i\mathbb{R}_\varepsilon$. This can also be done for all derivatives of f . We also note that the space defined above is homeomorphic to the space of functions f in $C^\infty(i\mathbb{R}_\varepsilon \times B)$ satisfying i), ii) and:

(v) For any polynomial P and any nonnegative integers M and N we have the estimate:

$$\sup_{\nu, k \in K} (1 + |\nu|)^N \left| \left(P \left(\frac{\partial}{\partial \nu} \right), \Omega_K^M \right) f(\nu, k) \right| < \infty.$$

Here we define the topology by the seminorms:

$$\tau_{P,M,N}^\varepsilon(f) = \sup_{\nu, k \in K} (1 + |\nu|)^N \left| \left(P \left(\frac{\partial}{\partial \nu} \right), \Omega_K^M \right) f(\nu, k) \right| < \infty,$$

that is, restriction to powers of the Casimir element does not alter the space or topology, see e.g. [Eg, p.193].

Let $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_e$ denote the space of functions $\psi \in \mathcal{S}(i\mathbb{R}_\varepsilon \times B)$ satisfying the symmetry condition (SC2):

$$\int_B e^{(-\nu+\rho)(A(x,b))} \psi(-\nu, b) db = \int_B e^{(\nu+\rho)(A(x,b))} \psi(\nu, b) db, \quad \nu \in i\mathbb{R}_\varepsilon, x \in X.$$

Lemma 2.2.4. *Let $\varepsilon \geq 0$.*

- (i) $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_e$ is a Fréchet space.
- (ii) $\mathcal{H}(\mathbb{C} \times B)$ is a dense subspace of $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)$.

Proof. (i) Clearly $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_e$ is a closed subspace of the Fréchet space $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)$, hence it is a Fréchet space in the inherited topology.

(ii) Let $\mathcal{S}(\mathbb{R} \times B)$ be the "classical" Schwartz space on $\mathbb{R} \times B$, i.e. the space of functions $f \in C^\infty(\mathbb{R} \times B)$ such that:

$$\sup_{t \in \mathbb{R}, k \in K} \left| (1 + |t|)^N \left(P \left(\frac{\partial}{\partial t} \right), E \right) f(t, k) \right| < \infty.$$

for any polynomial P , $E \in U(\mathfrak{k})$ and $N \in \mathbb{N}$. We consider the function $t \mapsto \cosh(\varepsilon \rho t)$. Let $\mathcal{S}_{\varepsilon \rho}(\mathbb{R} \times B)$ be the space of functions $f \in C^\infty(\mathbb{R} \times B)$ such that:

$$\sup_{t \in \mathbb{R}, k \in K} \left| (1 + |t|)^N \cosh(\varepsilon \rho t) \left(P \left(\frac{\partial}{\partial t} \right), E \right) f(t, k) \right| < \infty.$$

for any polynomial P , $E \in U(\mathfrak{k})$ and $N \in \mathbb{N}$. Then $\mathcal{S}_{\varepsilon\rho}(\mathbb{R} \times B)$ is a Fréchet space for the obvious topology. We now have two topological isomorphisms:

- 1) the map $f \mapsto \cosh(\varepsilon\rho)f$ between $\mathcal{S}_{\varepsilon\rho}(\mathbb{R} \times B)$ and the classical Schwartz space $\mathcal{S}(\mathbb{R} \times B)$. (obvious)
- 2) the "classical" Fourier transform \mathcal{F} between $\mathcal{S}_{\varepsilon\rho}(\mathbb{R} \times B)$ and $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)$.

Consider:

$$\hat{f}(\nu, b) = \int_{\mathbb{R}} f(t, b) e^{-\nu t} dt.$$

The factor $e^{-\nu t}$ is bounded by $2 \cosh(\varepsilon\rho t)$ for $|\operatorname{Re} \nu| \leq \varepsilon\rho$, hence we can write the above as:

$$\hat{f}(\nu, b) = \int_{\mathbb{R}} 2 \cosh(\varepsilon\rho t) f(t, b) \frac{1}{2 \cosh(\varepsilon\rho t)} e^{-\nu t} dt.$$

For fixed $k \in K$, we see by Morera's Theorem that $\hat{f}(\cdot, k)$ is holomorphic in $i\mathbb{R}_\varepsilon^o$. Now consider:

$$\begin{aligned} & \left| (1 + \nu)^N \left(P \left(\frac{\partial}{\partial \nu} \right), E \right) \hat{f}(\nu, k) \right| \\ &= \left| \int_{\mathbb{R}} \left\{ \left(1 + \left(\frac{d}{dt} \right) \right)^N P(-t) E f(t, k) \right\} e^{-\nu t} dt \right| \\ &\leq c \sup_{t \in \mathbb{R}, b \in B} \left| \cosh(\varepsilon\rho t) (1 + |t|)^2 \left(1 + \left(\frac{d}{dt} \right) \right)^N P(-t) E f(t, k) \right| \\ &\leq c \sup_{t \in \mathbb{R}, b \in B} \left| \cosh(\varepsilon\rho t) (1 + |t|)^M \left(\tilde{P} \left(\frac{\partial}{\partial t} \right), E \right) f(t, k) \right| < \infty, \end{aligned}$$

for some polynomial \tilde{P} and $M \in \mathbb{N}$, i.e., $\hat{f} \in \mathcal{S}(i\mathbb{R}_\varepsilon \times B)$ and the Fourier transform is continuous as an operator from $\mathcal{S}_{\varepsilon\rho}(\mathbb{R} \times B)$ to $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)$. Now consider the inverse Fourier transform:

$$\check{f}(t, k) = \frac{1}{2\pi} \int_{\mathbb{R}} f(i\nu, k) e^{i\nu t} d\nu.$$

Let P and Q be polynomials, then by Cauchy's Theorem, shifting integral from $i\nu$ to $i\nu \pm \varepsilon\rho$, we get:

$$\begin{aligned} & P(t) \cosh(\varepsilon\rho t) \left(Q \left(\frac{\partial}{\partial t} \right), E \right) \check{f}(t, k) = \\ & \frac{1}{4\pi} \int_{\mathbb{R}} P \left(i \frac{\partial}{\partial \nu} \right) \{ Q(i\nu - \varepsilon\rho) E f(i\nu - \varepsilon\rho, k) + Q(i\nu + \varepsilon\rho) E f(i\nu + \varepsilon\rho, k) \} e^{i\nu t} d\nu, \end{aligned}$$

i.e.,

$$\begin{aligned} & \sup_{t \in \mathbb{R}} \left| P(t) \cosh(\varepsilon\rho t) \left(Q \left(\frac{\partial}{\partial t} \right), E \right) \check{f}(t, k) \right| \\ & \leq c \sup_{\nu \in i\mathbb{R}_\varepsilon} \left| (1 + |\nu|)^2 P \left(\frac{-\partial}{\partial \nu} \right) Q(\nu) E f(\nu, k) \right| < \infty, \end{aligned}$$

and we see that the inverse Fourier transform also is continuous.

Since $C_c^\infty(\mathbb{R} \times B)$ is a dense subspace of $\mathcal{S}(\mathbb{R} \times B)$, we get from above that $C_c^\infty(\mathbb{R} \times B)$ is a dense subspace of $\mathcal{S}_{\varepsilon\rho}(\mathbb{R} \times B)$, and hence from the Paley-Wiener Theorem on $\mathbb{R} \times B$ we conclude ii). \square

Let in the following $\varepsilon \geq 0$. We want to show that $\mathcal{H}_e(\mathbb{C} \times B)$ is a dense subspace of $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_e$, that is, we have to look at functions in each space satisfying the symmetry conditions (SC1-2). We will make this problem a little easier by looking at K -types for the left-regular representation of K (l) on the Fréchet space $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)$ (acting on the second variable). It is straightforward to check that this representation is a smooth Fréchet representation of K . Let \hat{K} denote the set of equivalence classes of finite dimensional unitary irreducible representations (δ, V_δ) of K . Define $V_\delta^M \equiv \{v \in V_\delta | \delta(m)v = v, m \in M\}$ and $\hat{K}_M \equiv \{\delta \in \hat{K} | V_\delta^M \neq 0\}$. For $\delta \in \hat{K}$, let $d(\delta)$ and χ_δ denote the dimension and character of δ . Let $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_\delta$ be the closed subspace of $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)$ consisting of functions of K -type δ . The continuous projection of $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)$ onto $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_\delta$ is given by:

$$P_\delta f(\cdot, b) = f_\delta(\cdot, b) = d(\delta) \int_K \chi_\delta(k^{-1}) f(\cdot, k^{-1} \cdot b) dk.$$

[He1, Chapter IV, Lemma 1.7]. The space of K -finite functions in $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)$ is given as:

$$\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_K \equiv \{f \in \mathcal{S}(i\mathbb{R}_\varepsilon \times B) | \dim \text{span } l(K)f < \infty\}.$$

Then $f \in \mathcal{S}(i\mathbb{R}_\varepsilon \times B)_K \Leftrightarrow f = \sum_{\delta \in \hat{K}_M} f_\delta$, where the sum is finite. For $\delta \in \hat{K}$ we will also consider the contragredient representation $\check{\delta} \in \hat{K}$. $\check{\delta}(k)$ can be identified as the operator in $\text{Hom}(V'_\delta, V'_\delta)$ defined by $\check{\delta}(k) = \delta(k^{-1})^t$, where V'_δ is the dual space of V_δ and t denotes transpose. We remark that $\delta \in \hat{K}_M \Leftrightarrow \check{\delta} \in \hat{K}_M$. We have the following important result, see [He1, Chapter V, Theorem 3.1]:

Theorem 2.2.5. $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_K$ is a dense subspace of $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)$. Furthermore we have the expansion $f = \sum_{\delta \in \hat{K}_M} f_\delta = \sum_{\delta \in \hat{K}_M} f_{\check{\delta}}$, where the sums are absolutely convergent.

We easily see that $\mathcal{H}(\mathbb{C} \times B)$ is invariant under P_δ , hence we will consider the subspace $\mathcal{H}(\mathbb{C} \times B)_\delta = \mathcal{H}(\mathbb{C} \times B) \cap \mathcal{S}(i\mathbb{R}_\varepsilon \times B)_\delta$. Since P_δ is continuous we see that $\mathcal{H}(\mathbb{C} \times B)_\delta$ is a dense subspace of $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_\delta$. Denote by $\mathcal{H}(\mathbb{C} \times B)_{\delta,e}$, respectively by $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_{\delta,e}$, the set of functions in $\mathcal{H}(\mathbb{C} \times B)_\delta$ respectively in $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_\delta$ satisfying (SC2) (a function in $\mathcal{H}(\mathbb{C} \times B)_\delta$ that satisfies (SC2) will automatically by holomorphicity satisfy (SC1)). We want to show that $\mathcal{H}(\mathbb{C} \times B)_{\delta,e}$ is a dense subspace of $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_{\delta,e}$, for all $\delta \in \hat{K}_M$, and then by Theorem 2.2.5 conclude that $\mathcal{H}_e(\mathbb{C} \times B)$ is a dense subspace of $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_e$.

Let $\delta \in \hat{K}_M$ act on V_δ . Looking at matrix entries, we can define the spaces $\mathcal{H}(\mathbb{C} \times B, \text{Hom}(V_\delta, V_\delta))$ and $\mathcal{S}(i\mathbb{R}_\varepsilon \times B, \text{Hom}(V_\delta, V_\delta))$ of Paley-Wiener functions on $\mathbb{C} \times B$, respectively Schwartz functions on $i\mathbb{R}_\varepsilon \times B$, taking values in $\text{Hom}(V_\delta, V_\delta)$. We equip $\mathcal{S}(i\mathbb{R}_\varepsilon \times B, \text{Hom}(V_\delta, V_\delta))$ with the obvious topology. Define $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)^\delta \equiv \{F \in \mathcal{S}(i\mathbb{R}_\varepsilon \times B, \text{Hom}(V_\delta, V_\delta)) : F(\cdot, k \cdot b) = \delta(k)F(\cdot, b)\}$, and define $\mathcal{H}(\mathbb{C} \times B)^\delta = \mathcal{S}(i\mathbb{R}_\varepsilon \times B)^\delta \cap \mathcal{H}(\mathbb{C} \times B, \text{Hom}(V_\delta, V_\delta))$. For $f \in \mathcal{S}(i\mathbb{R}_\varepsilon \times B)$, we define:

$$P^\delta f(\cdot, b) = f^\delta(\cdot, b) = d(\delta) \int_K \delta(k) f(\cdot, k^{-1} \cdot b) dk.$$

We easily see that P^δ takes $\mathcal{H}(\mathbb{C} \times B)$ into $\mathcal{H}(\mathbb{C} \times B)^\delta$, and $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)$ into $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)^\delta$. Denote by $\mathcal{H}(\mathbb{C} \times B)_{e,\delta}^\delta$, respectively by $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_{e,\delta}^\delta$, the set of functions F in $\mathcal{H}(\mathbb{C} \times B)^\delta$ and $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)^\delta$ satisfying:

$$\int_B e^{(-\nu+\rho)(A(x,b))} F(-\nu, b) db = \int_B e^{(\nu+\rho)(A(x,b))} F(\nu, b) db, \quad \nu \in i\mathbb{R}_\varepsilon, x \in X.$$

Proposition 2.2.6. *Let $\delta \in \hat{K}_M$.*

(i) *The map:*

$$Q : F(\nu, b) \rightarrow \text{Tr}(F(\nu, b))$$

is a homeomorphism of $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)^\delta$ onto $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_{\delta, e}$, with inverse $f \rightarrow P^\delta f = f^\delta$. Furthermore Q takes $\mathcal{H}(\mathbb{C} \times B)^\delta$ onto $\mathcal{H}(\mathbb{C} \times B)_{\delta, e}$. Here Tr denote trace in $\text{Hom}(V_\delta, V_\delta)$.

(ii) *The image of $\mathcal{H}(\mathbb{C} \times B)_e^\delta$, respectively of $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_e^\delta$, under Q is $\mathcal{H}(\mathbb{C} \times B)_{\delta, e}$, respectively $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_{\delta, e}$.*

(iii) *The maps:*

$$P_\delta : \mathcal{S}(i\mathbb{R}_\varepsilon \times B) \rightarrow \mathcal{S}(i\mathbb{R}_\varepsilon \times B)_\delta \quad \text{and}$$

$$P^\delta : \mathcal{S}(i\mathbb{R}_\varepsilon \times B) \rightarrow \mathcal{S}(i\mathbb{R}_\varepsilon \times B)^\delta$$

are continuous open surjections, and the images are closed in $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)$ and $\mathcal{S}(i\mathbb{R}_\varepsilon \times B, \text{Hom}(V_\delta, V_\delta))$ respectively.

Proof. (i), (iii) As in [He1, p.395f].

(ii) Consider $f \in \mathcal{S}(i\mathbb{R}_\varepsilon \times B)_{\delta, e}$. In (SC2) replace x by $k^{-1} \cdot x$, use $A(k^{-1} \cdot x, b) = A(x, k \cdot b)$, make the substitution $b \mapsto k^{-1} \cdot b$, multiply by $\delta(k)$ and integrate over K . The result is:

$$\int_B e^{(-\nu+\rho)(A(x,b))} f^\delta(-\nu, b) db = \int_B e^{(\nu+\rho)(A(x,b))} f^\delta(\nu, b) db, \quad \nu \in i\mathbb{R}_\varepsilon, x \in X,$$

and thus P^δ takes $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_{\delta, e}$ into $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_e^\delta$. Taking trace we see that Q maps $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_e^\delta$ into $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_{\delta, e}$. \square

From Proposition 2.2.6(i), we see that $\mathcal{H}(\mathbb{C} \times B)^\delta$ is a dense subspace of $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)^\delta$. Furthermore, given $F \in \mathcal{S}(i\mathbb{R}_\varepsilon \times B)^\delta$, we can find $f \in \mathcal{S}(i\mathbb{R}_\varepsilon \times B)_{\delta, e}$ such that $F = f^\delta$. We will use this identification in the following. Consider $\delta \in \hat{K}_M$. Define the evaluation map:

$$(evf)(\nu) = f^\delta(\nu) \equiv f^\delta(\nu, eM).$$

We see that $\delta(m)f^\delta(\nu) = f^\delta(\nu)$, hence $f^\delta(\nu) \in \text{Hom}(V_\delta, V_\delta^M)$. The evaluation map is a homeomorphism between $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)^\delta$ and $\mathcal{S}(i\mathbb{R}_\varepsilon, \text{Hom}(V_\delta, V_\delta^M)) \equiv \mathcal{S}(i\mathbb{R}_\varepsilon)^\delta$, and between $\mathcal{H}(\mathbb{C} \times B)^\delta$ and $\mathcal{H}(\mathbb{C}, \text{Hom}(V_\delta, V_\delta^M)) \equiv \mathcal{H}(\mathbb{C})^\delta$. Lets consider $f \in \mathcal{S}(i\mathbb{R}_\varepsilon \times B)_e^\delta$. Then:

$$\begin{aligned} \int_B e^{(-\nu+\rho)(A(x,b))} f^\delta(-\nu, b) db &= \int_B e^{(\nu+\rho)(A(x,b))} f^\delta(\nu, b) db \Leftrightarrow \\ \int_K e^{(-\nu+\rho)(A(x,kM))} \delta(k) dk f^\delta(-\nu) &= \int_K e^{(\nu+\rho)(A(x,kM))} \delta(k) dk f^\delta(\nu) \Leftrightarrow \\ \Phi_{-\nu, \delta}(x) f^\delta(-\nu) &= \Phi_{\nu, \delta}(x) f^\delta(\nu), \end{aligned}$$

where $\Phi_{\nu, \delta}(x) = \int_K e^{(\nu+\rho)(A(x,kM))} \delta(k) dk$ is a generalized spherical function (or Eisenstein integral). We will consider the spaces $\mathcal{H}(\mathbb{C})_e^\delta$ in $\mathcal{H}(\mathbb{C})^\delta$ and $\mathcal{S}(i\mathbb{R}_\varepsilon)_e^\delta$ in $\mathcal{S}(i\mathbb{R}_\varepsilon)^\delta$ of functions satisfying:

$$\Phi_{-\nu, \delta}(x) f^\delta(-\nu) = \Phi_{\nu, \delta}(x) f^\delta(\nu), \quad \nu \in i\mathbb{R}_\varepsilon, x \in X, \quad (1)$$

and from above we see that we have homeomorphism between $\mathcal{H}(\mathbb{C} \times B)_e^\delta$ and $\mathcal{H}(\mathbb{C})_e^\delta$ and between $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_e^\delta$ and $\mathcal{S}(i\mathbb{R}_\varepsilon)_e^\delta$. Since G/K is a symmetric space of rank one, we see from [He2, Chapter II, Corollary 6.8] and [He1, Chapter V, Theorem 3.5] that $\dim V_\delta^M = 1$. Let v span V_δ^M and let $v_1, v_2, \dots, v_{d(\delta)}$ with $v_1 = v$ be an orthonormal basis of V_δ .

Lemma 2.2.7. *Let*

$$\varphi_{\nu,\delta}(x) = \langle \Phi_{\nu,\delta}(x)v, v \rangle = \int_K e^{(\nu+\rho)(A(x,kM))} \langle \delta(k)v, v \rangle dk,$$

and let

$$\varphi_{\nu,\delta}^j(x) = \langle \Phi_{\nu,\delta}(x)v, v_j \rangle = \int_K e^{(\nu+\rho)(A(x,kM))} \langle \delta(k)v, v_j \rangle dk, \quad 1 \leq j \leq d(\delta).$$

Then $\varphi_{\nu,\delta}^j(ka \cdot x_o) = \langle \delta(k)v, v_j \rangle \varphi_{\nu,\delta}(a \cdot x_o)$, where $x_o = eK$, $k \in K$, $a \in A$.

Proof. Define $F : X \rightarrow V_\delta$ by:

$$F(x) = \int_K e^{(\nu+\rho)(A(x,kM))} \delta(k)v dk = \Phi_{\nu,\delta}(x)v.$$

Then $\varphi_{\nu,\delta}^j(x) = \langle F(x), v_j \rangle$ and since $F(k \cdot x) = \delta(k)F(x)$ we have $F(a \cdot x_o) \in V_\delta^M$ ($\delta(m)F(a \cdot x_o) = F(am \cdot x_o) = F(a \cdot x_o)$, $m \in M = Z_K(A)$). Since $\dim V_\delta^M = 1$ we deduce: $F(a \cdot x_o) = \varphi_{\nu,\delta}(a \cdot x_o)v$. Then:

$$\varphi_{\nu,\delta}^j(ka \cdot x_o) = \langle F(ka \cdot x_o), v_j \rangle = \langle \delta(k)F(a \cdot x_o), v_j \rangle = \langle \delta(k)v, v_j \rangle \varphi_{\nu,\delta}(a \cdot x_o).$$

□

Since $f^\delta(\nu) \in \text{Hom}(V_\delta, V_\delta^M)$ we see from Lemma 2.2.7 that (1) is equivalent to:

$$\varphi_{-\nu,\delta}(a \cdot x_o) f^\delta(-\nu) = \varphi_{\nu,\delta}(a \cdot x_o) f^\delta(\nu), \quad \nu \in i\mathbb{R}_\varepsilon. \quad (2)$$

We can determine $\varphi_{\nu,\delta}$ quite explicitly in terms of the hypergeometric functions (See [He2, Chapter III, Theorem 11.2]). As a corollary we get:

Lemma 2.2.8. *The functions $\varphi_{\nu,\delta}$ satisfies the symmetry condition:*

$$\varphi_{\nu,\delta}(a \cdot x_o) = \varphi_{-\nu,\delta}(a \cdot x_o) \frac{p_\delta(\nu)}{p_\delta(-\nu)},$$

where $p_\delta(\nu)$ is a polynomial of the form:

$$p_\delta(\nu) = (\nu + \rho + s - 1) \cdots (\nu + \rho), \quad s \in \mathbb{N} \quad \text{or} \quad p_\delta(\nu) \equiv 1.$$

Proof. [He2, Chapter III, Corollary 11.3].

□

So (2) is equivalent to:

$$p_\delta(-\nu) f^\delta(-\nu) = p_\delta(\nu) f^\delta(\nu), \quad \nu \in i\mathbb{R}_\varepsilon. \quad (3)$$

Consider the set of even functions in $\mathcal{H}(\mathbb{C})^\delta$, $\mathcal{H}(\mathbb{C})_1^\delta$ and the set of even functions in $\mathcal{S}(i\mathbb{R}_\varepsilon)^\delta$, $\mathcal{S}(i\mathbb{R}_\varepsilon)_1^\delta$. Then we have the following lemma:

Lemma 2.2.9. *The map:*

$$G(\nu) \rightarrow F(\nu) = p_\delta(-\nu)G(\nu)$$

is a homeomorphism of $\mathcal{S}(i\mathbb{R}_\varepsilon)_1^\delta$ onto $\mathcal{S}(i\mathbb{R}_\varepsilon)_e^\delta$ taking $\mathcal{H}(\mathbb{C})_1^\delta$ to $\mathcal{H}(\mathbb{C})_e^\delta$.

Proof. The map clearly is continuous, and it takes $\mathcal{H}(\mathbb{C})_1^\delta$ into $\mathcal{H}(\mathbb{C})_e^\delta$ and $\mathcal{S}(i\mathbb{R}_\varepsilon)_1^\delta$ into $\mathcal{S}(i\mathbb{R}_\varepsilon)_e^\delta$. Since p_δ is a nonzero polynomial we see that the map is injective. Let $F \in \mathcal{S}(i\mathbb{R}_\varepsilon)_e^\delta$, and consider the function:

$$G(\nu) = \frac{F(\nu)}{p_\delta(-\nu)}.$$

G is even:

$$G(-\nu) = \frac{F(-\nu)}{p_m(\nu)} \cdot \frac{p_m(-\nu)}{p_m(-\nu)} = \frac{F(\nu)}{p_m(\nu)} \cdot \frac{p_m(\nu)}{p_m(-\nu)} = \frac{F(\nu)}{p_m(-\nu)} = G(\nu).$$

Since ν and $-\nu$ are not both roots for p_δ we see that one of the expressions $G(\nu) = \frac{F(\nu)}{p_m(-\nu)} = \frac{F(-\nu)}{p_m(\nu)}$ always will be welldefined, and then G will satisfy the same differentiability and growth conditions as F . Hence the map is surjective, and by the closed graph Theorem the map is a homeomorphism with the required properties. \square

Via the classical Fourier transform it is easy to verify that $\mathcal{H}(\mathbb{C})_1^\delta$ is dense in $\mathcal{S}(i\mathbb{R}_\varepsilon)_1^\delta$, which yields that $\mathcal{H}(\mathbb{C})_e^\delta$ is dense in $\mathcal{S}(i\mathbb{R}_\varepsilon)_e^\delta$, and thus we get the desired result:

Theorem 2.2.10. *Let $\varepsilon \geq 0$ and let $\delta \in \hat{K}_M$, then:*

- (i) $\mathcal{H}(\mathbb{C})_e^\delta$ is dense in $\mathcal{S}(i\mathbb{R}_\varepsilon)_e^\delta$.
- (ii) $\mathcal{H}(\mathbb{C} \times B)_{\delta,e}$ is a dense subspace of $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_{\delta,e}$.
- (iii) $\mathcal{H}_e(\mathbb{C} \times B)$ is a dense subspace of $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_e$.

Proof. (ii) Since $\mathcal{H}(\mathbb{C} \times B)_{\delta,e}$ is dense in $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_{\delta,e}$ for all $\delta \in \hat{K}_M$.

(iii) We have $f \in \mathcal{S}(i\mathbb{R}_\varepsilon \times B)_e \Leftrightarrow f_\delta \in \mathcal{S}(i\mathbb{R}_\varepsilon \times B)_{\delta,e} \forall \delta \in \hat{K}_M$.

\Leftarrow : $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_e$ is closed in $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)$.

\Rightarrow : In (SC2) replace x by $k^{-1} \cdot x$, use $A(k^{-1} \cdot x, b) = A(x, k \cdot b)$, make the substitution $b \mapsto k^{-1} \cdot b$, multiply by $\chi_\delta(k^{-1})$ and integrate over K . The result is:

$$\int_B e^{(-\nu+\rho)(A(x,b))} f_\delta(-\nu, b) db = \int_B e^{(\nu+\rho)(A(x,b))} f_\delta(\nu, b) db.$$

which exactly means that $f_\delta \in \mathcal{S}(i\mathbb{R}_\varepsilon \times B)_{\delta,e}$.

By Theorem 2.2.10ii), every finite sum of the type $\sum_{\delta \in \hat{K}_M} f_\delta$, $f_\delta \in \mathcal{S}(i\mathbb{R}_\varepsilon \times B)_{\delta,e}$, can be approximated by a finite sum $\sum_{\delta \in \hat{K}_M} g_\delta$, $g_\delta \in \mathcal{H}(\mathbb{C} \times B)_{\delta,e}$, and the theorem then follows by Theorem 2.2.5. \square

Remark: As the primary goal was to show Theorem 1 via Anker's method, Theorem 1.2.10iii) is very much an important result. But since we can only handle the K -finite case, the important result is actually Theorem 1.2.10i).

Later we will need some estimates of the matrix coefficients of the generalized spherical functions (as for spherical functions in [An]). Let $P = MAN$ be the minimal parabolic subgroup of G . Consider the characters $\{ma_t n \mapsto e^{\nu t}, \nu \in \mathbb{C}\}$ of P . The spherical principal series representations π_ν on G are then the induced representations coming from these characters. They are realized on $L^2(K/M)$ via the formula:

$$\{\pi_\nu(g)f\}(k) = e^{-(\nu+\rho)H(g^{-1}k)} f(\kappa(g^{-1}k)), \quad f \in L^2(K/M).$$

The restriction of π_ν to K is: $\{\pi_\nu(k_1)f\}(k_2) = f(k_1^{-1}k_2)$.

Proposition 2.2.11. *Let $\phi \in C^\infty(K/M)$, and consider the function:*

$$\psi_\nu(g) = \int_K \phi(k) e^{-(\nu+\rho)(H(g^{-1}k))} dk.$$

Given $D, E \in U(\mathfrak{g})$, there is a constant c and elements $D_l \in U(\mathfrak{k})$, $1 \leq l \leq M$ such that:

$$|\psi_\nu(D; g; E)| < c(1 + |\nu|)^{\deg D + \deg E} \varphi_{\text{Re}\nu}(g) \left\{ \sum_{l=1}^M \sup_{k \in K} |\phi(k; D_l)| \right\}.$$

Proof. We can write $|\psi_\nu(D; g; E)|$ as:

$$\begin{aligned} \left| \int_K \phi(k) \pi_\nu(D; g; E) 1_{K/M} dk \right| &= \left| \int_K \phi(k) \pi_\nu(D) \pi_\nu(g) \pi_\nu(E) 1_{K/M} dk \right| \\ &= |\langle \pi_\nu(g) \pi_\nu(E) 1_{K/M}, \pi_{\bar{\nu}}(D^*) \overline{\phi(k)} \rangle_2| \\ &\leq \|\pi_\nu(g) \pi_\nu(E) 1_{K/M}\|_1 \|\pi_{\bar{\nu}}(D^*) \overline{\phi}\|_\infty, \end{aligned}$$

by the Hölder inequality. $\pi_\nu(E) 1_{K/M}$ can be considered as a function on $K/M : \xi$, hence we have:

$$[\pi_\nu(g) \xi](k) = e^{-(\nu+\rho)H(g^{-1}k)} \xi(\kappa(g^{-1}k)).$$

Again using Hölder this gives us:

$$\|\pi_\nu(g) \pi_\nu(E) 1_{K/M}\|_1 \leq \|\xi(k)\|_\infty \|\pi_\nu(g) 1_{K/M}\|_1 = \|\pi_\nu(E) 1_{K/M}\|_\infty \varphi_{\text{Re}\nu}(g).$$

We are left by estimating the two norms:

- (i) $\|\pi_\nu(E) 1_{K/M}\|_\infty$
- (ii) $\|\pi_{\bar{\nu}}(D^*) \phi(k)\|_\infty$

From [An, p.336] we get an estimate on (i) $\|\pi_\nu(E) 1_{K/M}\|_\infty \leq c(1 + |\nu|)^{\deg E}$. So consider (ii). Introduce the auxiliary function $f_\nu(g) = e^{-(\nu+\rho)H(g)}$, then by the Leibniz rule of differentiation:

$$\begin{aligned} \pi_\nu(D^*) \phi(k) &= \sum_{\deg D = \deg D' + \deg D''} f_\nu(D'; k) \phi(\kappa(D''; k)) \\ &= \sum_{\deg D = \deg D' + \deg D''} f_\nu(D'; k) \phi(\kappa(k; \text{Ad}(k^{-1})D'')). \end{aligned}$$

We see that $f_\nu(D'; k) = \{\pi_\nu(D'^*) 1_{K/M}\}(k)$. There exists functions ξ_1, \dots, ξ_m in $C^\infty(K)$ and elements D_1, \dots, D_m in $U(\mathfrak{g})$ of degree $\leq \deg D$ such that $\text{Ad}(k^{-1})D'' = \sum_{l=1}^m \xi_l(k) D_l$. By the PBW-Theorem we can write:

$$U(\mathfrak{g}) = U(\mathfrak{k}) \oplus U(\mathfrak{g})(\mathfrak{a} + \mathfrak{n}).$$

Let D'_l be the projection of D_l on $U(\mathfrak{k})$. Then:

$$\begin{aligned} |\phi(\kappa(k; \text{Ad}(k^{-1})D''))| &= |\phi(\kappa(k; \sum_{l=1}^m \xi_l(k) D_l))| = \left| \sum_{l=1}^m \xi_l(k) \phi(\kappa(k; D_l)) \right| \\ &= \left| \sum_{l=1}^m \xi_l(k) \phi(\kappa(k; D'_l)) \right| \leq c \sum_{l=1}^m \sup_{k \in K} |\phi(k; D'_l)|, \end{aligned}$$

and thus:

$$\|\pi_{\bar{\nu}}(D^*)\phi\|_{\infty} \leq c(1 + |\nu|)^{\deg D} \sum_{\deg D'' \leq \deg D} \sum_{l=1}^m \sup_{k \in K} |\phi(k; D'_l)|.$$

□

2.3. The isomorphism of the Fourier transform on K -finite elements in the L^p -Schwartz spaces.

In this section we show the generalization of [An] in the rank 1 case. Let $f \in \mathcal{S}^p(G/K)$, $0 < p \leq 2$. Since $C_c^{\infty}(G/K) \subset \mathcal{S}^p(G/K) \subset L^2(G/K)$, we can define the Fourier transform $\hat{f} \in L^2(i\mathbb{R}_+ \times B, c|c(i\nu)|^{-2})$. From the definition of $\mathcal{S}^p(G/K)$, and the Cartan decomposition of the measure on X , we see that the extension of the Fourier transform from $C_c^{\infty}(G/K)$ to $\mathcal{S}^p(G/K)$ is trivial, that is:

$$\mathcal{H}f(\nu, b) = \hat{f}(\nu, b) = \int_X f(x) e^{(-\nu + \rho)(A(x, b))} dx,$$

for all $\nu \in i\mathbb{R}$, $b \in B$. Furthermore we see that the above integral is welldefined for $(\nu, b) \in i\mathbb{R}_{\varepsilon} \times B$. We actually have:

Lemma 2.3.1. *Let $f \in \mathcal{S}^p(G/K)$, and let $\varepsilon = \frac{2}{p} - 1$. Then $\hat{f} \in C^{\infty}(i\mathbb{R}_{\varepsilon} \times B)$, and $\hat{f}(\cdot, b)$ is holomorphic in $i\mathbb{R}_{\varepsilon}^o$ for fixed $b \in B$.*

Proof. Let

$$\hat{f}(\nu, b) = \int_X f(x) e^{(-\nu + \rho)(A(x, b))} dx.$$

By the Cartan decomposition we have:

$$\hat{f}(\nu, b) = \int_0^{\infty} \int_K f(ka_t) e^{(-\nu + \rho)(A(ka_t x_o, b))} dk \sinh^n t dt.$$

Consider the above integral over K :

$$\begin{aligned} \left| \int_K f(ka_t) e^{(-\nu + \rho)(A(ka_t x_o, b))} dk \right| &\leq \left\{ \sup_{k \in K} |f(ka_t)| \right\} \int_K e^{(-\operatorname{Re} \nu + \rho)(A(ka_t x_o, b))} dk \\ &= \left\{ \sup_{k \in K} |f(ka_t)| \right\} \varphi_{\operatorname{Re} \nu}(a_t). \end{aligned}$$

We then see:

- 1) $e^{-2(\frac{1}{p}-1)\rho t} (1+t)^{-1} \varphi_{\operatorname{Re} \nu}(a_t)$ is a bounded function for $\nu \in i\mathbb{R}_{\varepsilon}^o$ (by Lemma 2.1.9).
- 2) $t \mapsto \{\sup_{k \in K} |f(ka_t)|\} \in C(\overline{\mathbb{R}_+})$.
- 3) $\sup_{t > 0} (1+t)^N \varphi_o(a_t)^{-\frac{2}{p}} \{\sup_{k \in K} |f(ka_t)|\} \leq \sup_{g \in G} (1+|g|)^N \varphi_o(g)^{-\frac{2}{p}} |f(g)| < \infty$.
- 4) $t \mapsto e^{2(\frac{1}{p}-1)\rho t} (1+t) \{\sup_{k \in K} |f(ka_t)|\} \in L^1(\mathbb{R}_+, \sinh^n t dt)$ (Lemma 2.1.9).

We thus see that:

- 5) $\hat{f}(\nu, b)$ is welldefined for all $\nu \in i\mathbb{R}_{\varepsilon}^o$, $b \in B$.
- 6) By Morera's Theorem we see that $\hat{f}(\cdot, b)$ is holomorphic in $i\mathbb{R}_{\varepsilon}^o$. Differentiability follows from Lebesgue's Dominated Convergence Theorem. □

Theorem 2.3.2. *Let $0 < p \leq 2$, $\varepsilon = \frac{2}{p} - 1$. The Fourier transform \mathcal{H} is an injective and continuous homomorphism from $\mathcal{S}^p(G/K)$ into $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_e$.*

Proof. Let $f \in \mathcal{S}^p(G/K)$. By Lemma 2.3.1 we see that $\mathcal{H}f(\cdot, b)$ is holomorphic in $i\mathbb{R}_\varepsilon^o$ for fixed $b \in B$. Now we observe that $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_e$ and its topology also is determined by the set of seminorms:

$$\tilde{\tau}_{P,M,N}^\varepsilon(f) = \sup_{\nu \in i\mathbb{R}_\varepsilon, k \in K} \left| \left(P \left(\frac{\partial}{\partial \nu} \right), \Omega_K^M \right) \{(\nu^2 - \rho^2 + d)^N f(\nu, k)\} \right|,$$

where P , M and N are as before, and d is a constant such that $\nu^2 - \rho^2 + d \neq 0$. Fix $\tilde{\tau}_{P,M,N}^\varepsilon$. Then

$$\begin{aligned} & \left(P \left(\frac{\partial}{\partial \nu} \right), \Omega_K^M \right) (\nu^2 - \rho^2 + d)^N \mathcal{H}f(\nu, k) \\ &= \left(P \left(\frac{\partial}{\partial \nu} \right), \Omega_K^M \right) (\nu^2 - \rho^2 + d)^N \int_X f(x) e^{(-\nu + \rho)(A(x, kM))} dx \\ &= \int_X (\Omega_K^M (\Delta + d)^N f)(x) P \left(\frac{\partial}{\partial \nu} \right) e^{(-\nu + \rho)(A(x, kM))} dx, \end{aligned}$$

where $\Omega_K^M f(x) = \Omega_K^M f(g) = f(\Omega_K^M g)$, $x = gK$. Now:

$$\begin{aligned} \left| P \left(\frac{\partial}{\partial \nu} \right) e^{(-\nu + \rho)(A(x, kM))} \right| &= \left| P(-A(x, kM)) e^{(-\nu + \rho)(A(x, kM))} \right| \\ &\leq c(1 + |x|)^{\deg P} e^{(-Re\nu + \rho)(A(x, kM))}. \end{aligned}$$

As before (Lemma 2.3.1), we use the Cartan decomposition and the estimate $\sinh^n t \leq e^{2\rho t}$:

$$\begin{aligned} & \int_X \left| (\Omega_K^M (\Delta + d)^N f)(x) P \left(\frac{\partial}{\partial \nu} \right) e^{(-\nu + \rho)(A(x, kM))} \right| dx \leq \\ & c \int_0^\infty \left\{ \sup_{k' \in K} |(\Omega_K^M (\Delta + d)^N f)(k' a_t)| \right\} (1+t)^{\deg P} \int_K e^{(-Re\nu + \rho)(A(k' a_t, kM))} dk' e^{2\rho t} dt \\ &= c \int_0^\infty \left\{ \sup_{k \in K} |(\Omega_K^M (\Delta + d)^N f)(ka_t)| \right\} (1+t)^{\deg P} \varphi_{Re\nu}(a_t) e^{2\rho t} dt. \end{aligned}$$

By Lemma 2.1.9 ii) there exists a constant c such that $|\varphi_{Re\nu}(a_t)| \leq c(1+t)e^{(\frac{2}{p}-2)\rho t}$, for $t > 0$. Let

$$\sigma(f) = \sup_{g \in G} (1 + |g|)^{M'} \varphi_o(g)^{-\frac{2}{p}} |(\Omega_K^M (\Delta + d)^N f)(g)|,$$

where we choose $M' \in \mathbb{N}$ s.t. $M' > \deg P + 1 + \frac{2}{p}$. Then from Lemma 2.1.9 i) we get:

$$|(\Omega_K^M (\Delta + d)^N f)(g)| \leq c\sigma(f)(1 + |g|)^{-M' - 1 + \frac{2}{p}} e^{-\frac{2}{p}\rho|g|}.$$

Putting this together we get:

$$\tilde{\tau}_{P,M,N}^\varepsilon(\mathcal{H}f) \leq c\sigma(f)$$

and σ is obviously bounded by a sum of seminorms for $\mathcal{S}^p(G/K)$, and we have shown continuity.

Injectivity follows from Theorem 2.1.3. \square

Note, that the restriction to powers of Ω_K is not crucial, we could prove the theorem for the topology with Ω_K replaced by $E \in U(\mathfrak{g})$ in the same fashion.

Now, of course, we would like to show surjectivity of \mathcal{H} . In order to generalize the proof from [An], we need a cut-off function, ω_j , w.r.t. to the first variable of $H(\cdot, \cdot)$, in order to control the involved functions in the commutative diagram below:

$$\begin{array}{ccc}
 & h \in \mathcal{H}_e(\mathbb{C} \times B) & \\
 \mathcal{H} \nearrow & & \nwarrow \mathcal{F} \\
 f \in C_c^\infty(G/K) & \xrightarrow[\cong]{\mathcal{R}} & H \in \mathcal{RC}_c^\infty(G/K)
 \end{array}$$

Unfortunately the symmetry conditions imposed on h and H are rather obscure, as we have already encountered, therefore looking at the function H decomposed w.r.t. to the cut-off function,

$$H(t, b) = \omega_j(t)H(t, b) + (1 - \omega_j(t))H(t, b), \quad H_j(t, b) = (1 - \omega_j(t))H(t, b),$$

will not necessarily give us a function H_j in $\mathcal{RC}_c^\infty(G/K)$. This means that we will be unable to define functions $h_j = \mathcal{F}H_j \in \mathcal{H}_e(\mathbb{C} \times B)$ and $f_j = \mathcal{R}^{-1}H_j \in C_c^\infty(G/K)$, as e.g., f_j , though welldefined, doesn't need to have compact support. A way to avoid the difficulties created by the symmetry conditions is to look at fixed K -types, as we did with the density argument.

So consider the left-regular representation of K (l) on the Fréchet space $\mathcal{S}^p(G/K)$. It is straightforward to check that this representation is a smooth Fréchet representation of K . With $\delta \in \hat{K}_M$ acting on V_δ , consider the spaces $C_c^\infty(G/K, \text{Hom}(V_\delta, V_\delta))$ and $\mathcal{S}^p(G/K, \text{Hom}(V_\delta, V_\delta))$ of compactly supported differentiable functions on G/K , respectively L^p -Schwartz functions on G/K , taking values in $\text{Hom}(V_\delta, V_\delta)$ (looking at the matrix-entries). Define $\mathcal{S}^p(G/K)^\delta \equiv \{F \in \mathcal{S}^p(G/K, \text{Hom}(V_\delta, V_\delta)) : F(k \cdot x) = \delta(k)F(x)\}$, and define $C_c^\infty(G/K)^\delta = \mathcal{S}^p(G/K)^\delta \cap C_c^\infty(G/K, \text{Hom}(V_\delta, V_\delta))$. For $f \in \mathcal{S}^p(G/K)$ we define:

$$'P^\delta f(x) = f^\delta(x) = d(\delta) \int_K \delta(k) f(k^{-1} \cdot x) dk.$$

We see that $'P^\delta$ takes $C_c^\infty(G/K)$ into $C_c^\infty(G/K)^\delta$, and $\mathcal{S}^p(G/K)$ into $\mathcal{S}^p(G/K)^\delta$. Let furthermore $C_c^\infty(G/K)_\delta$ and $\mathcal{S}^p(G/K)_\delta$ denote the space of K -finite functions in $C_c^\infty(G/K)$, respectively in $\mathcal{S}^p(G/K)$, of type δ . The continuous projection of $C_c^\infty(G/K)$, respectively of $\mathcal{S}^p(G/K)$, onto $C_c^\infty(G/K)_\delta$ and $\mathcal{S}^p(G/K)_\delta$ is given by:

$$'P_\delta f(x) = f_\delta(x) = d(\delta) \int_K \chi_\delta(k^{-1}) f(k^{-1} \cdot x) dk.$$

Proposition 2.3.3. *Let $\delta \in \hat{K}_M$.*

(i) *The map:*

$$Q : F(x) \rightarrow \text{Tr}(F(x))$$

is a homeomorphism of $\mathcal{S}^p(G/K)^\delta$ onto $\mathcal{S}^p(G/K)_\delta$, with inverse $f \rightarrow 'P^\delta f = f^\delta$. Furthermore Q takes $C_c^\infty(G/K)^\delta$ onto $C_c^\infty(G/K)_\delta$. Here Tr denotes trace in $\text{Hom}(V_\delta, V_\delta)$.

(ii) *The maps:*

$$\begin{aligned}
 'P_\delta : \mathcal{S}^p(G/K) &\rightarrow \mathcal{S}^p(G/K)_\delta, \quad \text{and} \\
 'P^\delta : \mathcal{S}^p(G/K) &\rightarrow \mathcal{S}^p(G/K)^\delta,
 \end{aligned}$$

are continuous open surjections, and the images are closed in $\mathcal{S}^p(G/K)$, respectively in $\mathcal{S}^p(G/K, \text{Hom}(V_\delta, V_\delta))$.

Proof. As for proposition 2.2.6. □

Definition 2.3.4. Let $\delta \in \hat{K}_M$. For $f \in \mathcal{S}^p(G/K)_\delta$ the δ -spherical transform is defined by:

$$\begin{aligned} \mathcal{H}^\delta f(\nu) &= (ev \circ P^\delta \mathcal{H}f)(\nu) \\ &= d(\delta) \int_K \int_X f(x) e^{(-\nu+\rho)(A(x,kM))} dx \delta(k^{-1}) dk \\ &= d(\delta) \int_X f(x) \int_K e^{(-\nu+\rho)(A(x,kM))} \delta(k^{-1}) dk dx \\ &= d(\delta) \int_X f(x) \Phi_{-\bar{\nu}, \delta}(x)^* dx, \end{aligned}$$

where: $\Phi_{-\bar{\nu}, \delta}(x)^* = \int_K e^{(-\nu+\rho)(A(x,kM))} \delta(k^{-1}) dk$ is the adjoint of the generalized spherical function $\Phi_{-\bar{\nu}, \delta}$, $*$ denoting the adjoint in $\text{Hom}(V_\delta, V_\delta)$, and ev is the evaluation map, $evf(\nu) = f(\nu, eM)$.

For the trivial representation we have $\Phi_{-\bar{\nu}, 1}(x)^* = \varphi_\nu(x)$, that is, the 1-spherical transform \mathcal{H}^1 is the "classical" spherical transform \mathcal{H} . For the δ -spherical transform we have the following Paley-Wiener Theorem:

Theorem 2.3.5. The δ -spherical transform $f \mapsto \mathcal{H}^\delta f$ is a bijection of $C_c^\infty(G/K)_\delta$ onto $\mathcal{H}(\mathbb{C})_e^\delta$.

Proof. It is evident from the definition that \mathcal{H}^δ maps $C_c^\infty(G/K)_\delta$ into $\mathcal{H}(\mathbb{C})_e^\delta$, and also that \mathcal{H}^δ is injective since \mathcal{H} is an injective map. For surjectivity, let $\psi \in \mathcal{H}(\mathbb{C})_e^\delta$. The function $\Psi(\nu, kM) \equiv \text{Tr}(\delta(k)\psi(\nu))$ is clearly a holomorphic function of uniform exponential type on $\mathbb{C} \times B$. By Proposition 2.2.6 and the discussion afterwards, we see that $\Psi \in \mathcal{H}(\mathbb{C} \times B)_\delta$. By the Paley-Wiener Theorem 2.1.5, there exists a unique $F \in C_c^\infty(G/K)$ such that $\Psi = \mathcal{H}F$. By Proposition 2.3.3 the function: $'P_\delta F = F_\delta$ belongs to $C_c^\infty(G/K)_\delta$, and we have:

$$\begin{aligned} \mathcal{H}^\delta F_\delta(\nu) &= \{P^\delta \mathcal{H}' P_\delta F\}(\nu, eM) = \{P^\delta P_\delta \mathcal{H}F\}(\nu, eM) = \{P^\delta P_\delta \Psi\}(\nu, eM) \\ &= d^2(\delta) \int_K \int_K \Psi(\nu, u^{-1}kM) \chi_\delta(u) \delta(k^{-1}) dk du \\ &= d^2(\delta) \int_K \int_K \text{Tr}(\delta(k)\psi(\nu)) \chi_\delta(u) \delta(k^{-1}u^{-1}) dk du \\ &= d^2(\delta) \int_K \int_K \text{Tr}(\delta(k)\psi(\nu)) \chi_\delta(u) \delta(k^{-1}) \delta(u^{-1}) dk du \\ &= d^2(\delta) \int_K \delta(k^{-1}) \text{Tr}(\delta(k)\psi(\nu)) dk \int_K \chi_\delta(u) \delta(u^{-1}) du \\ &= d(\delta) \int_K \delta(k^{-1}) \text{Tr}(\delta(k)\psi(\nu)) dk \\ &= P^\delta \text{Tr}(\delta(k)\psi(\nu)) \\ &= \psi(\nu), \end{aligned}$$

since the orthogonality relations yields:

$$d(\delta) \int_K \bar{\delta}_{i,j}(u) \delta(u) du = E_{i,j}.$$

□

We also have the following inversion Theorem:

Theorem 2.3.6. *The δ -spherical transform is inverted by:*

$$f(x) = c \operatorname{Tr} \left\{ \int_0^\infty \Phi_{i\nu, \delta}(x) (\mathcal{H}^\delta f)(i\nu) |c(i\nu)|^{-2} d\nu \right\}, \quad f \in C_c^\infty(G/K)_\delta,$$

where

$$\Phi_{i\nu, \delta}(x) = \int_K e^{(i\nu + \rho)(A(x, kM))} \delta(k) dk.$$

Proof. If $f \in C_c^\infty(G/K)_\delta$, then:

$$\begin{aligned} \mathcal{H}f(\nu, kM) &= \mathcal{H}'P_\delta f(\nu, kM) = P_\delta \mathcal{H}f(\nu, kM) \\ &= \operatorname{Tr} P^\delta P_\delta \mathcal{H}f(\nu, kM) = \operatorname{Tr} P^\delta \mathcal{H}'P_\delta f(\nu, kM) \\ &= \operatorname{Tr} \mathcal{H}^\delta f(\nu, kM) = \operatorname{Tr}(\delta(k) \mathcal{H}^\delta f(\nu)). \end{aligned}$$

So by the "classical" inversion Theorem, Theorem 2.1.2, we get:

$$\begin{aligned} f(x) &= c \int_{\mathbb{R}_+ \times K} e^{(i\nu + \rho)(A(x, kM))} \operatorname{Tr}(\delta(k) \mathcal{H}^\delta f(i\nu)) |c(i\nu)|^{-2} d\nu \\ &= c \operatorname{Tr} \left\{ \int_{\mathbb{R}_+ \times K} e^{(i\nu + \rho)(A(x, kM))} \delta(k) \mathcal{H}^\delta f(i\nu) |c(i\nu)|^{-2} d\nu \right\}. \end{aligned}$$

□

Let v span V_δ^M , and let $v_1, v_2, \dots, v_{d(\delta)}$, with $v_1 = v$, be an orthonormal basis of V_δ . Then the inverse δ -spherical transform can be written as:

$$\begin{aligned} f(x) &= c \int_0^\infty \operatorname{Tr}(\Phi_{i\nu, \delta}(x) \mathcal{H}^\delta f(i\nu)) |c(i\nu)|^{-2} d\nu \\ &= c \int_0^\infty \sum_{l=1}^{d(\delta)} \Phi_{i\nu, \delta}(x)_{l,1} \mathcal{H}^\delta f(i\nu)_{1,l} |c(i\nu)|^{-2} d\nu, \end{aligned}$$

where i, j denote matrix entries. We see that $\Phi_{i\nu, \delta}(x)_{l,1} = \varphi_{i\nu, \delta}^l(x)$ (see Lemma 2.2.7).

Moreover we also have a Plancherel Theorem:

Theorem 2.3.7.

$$\begin{aligned} \int_X |f(x)|^2 dx &= \int_0^\infty \operatorname{Tr} \{ (\mathcal{H}^\delta f)(i\nu) (\mathcal{H}^\delta f)^*(i\nu) \} |c(i\nu)|^{-2} d\nu \\ &= c \int_0^\infty \| \{ (\mathcal{H}^\delta f)(i\nu) \} \|_{HS}^2 |c(i\nu)|^{-2} d\nu. \end{aligned}$$

Proof. As above. □

Consider the classical Fourier transform \mathcal{F} acting on vector valued functions, and define $C_c^\infty(\mathbb{R})_e^\delta = \mathcal{F}^{-1} \mathcal{H}(\mathbb{C})_e^\delta$, that is, functions in $C_c^\infty(\mathbb{R}, \operatorname{Hom}(V_\delta, V_\delta))$ satisfying certain symmetry conditions, then we have the following commutative diagram:

$$\begin{array}{ccc} C_c^\infty(G/K)_\delta & \xrightarrow{\mathcal{H}^\delta} & \mathcal{H}(\mathbb{C})_e^\delta \\ \mathcal{R} \downarrow & & \uparrow \mathcal{F} \\ C_c^\infty(\mathbb{R} \times B)_{\delta, e} & \xrightarrow{ev \circ P^\delta} & C_c^\infty(\mathbb{R})_e^\delta \end{array}$$

where:

$${}''P^\delta f(t, b) = d(\delta) \int_K \delta(k) f(t, k^{-1} \cdot b) dk,$$

and \mathcal{R} is the Radon transform. We define a new transform \mathcal{T} by:

$$\mathcal{T} = ev \circ {}''P^\delta \mathcal{R}.$$

More exactly we have:

$$\mathcal{T}f(t) = e^{\rho t} \int_{K \times N} f(ka_t n) \delta(k^{-1}) dk dn,$$

which can be thought of as a generalized Abel transform.

Theorem 2.3.8. *The transform \mathcal{T} is an isomorphism between $C_c^\infty(G/K)_\delta$ and $C_c^\infty(\mathbb{R})_e^\delta$. Moreover $\text{supp} f \subset \bar{B}(0, R)$ if and only if $\text{supp} \mathcal{T}f \subset \bar{B}(0, R)$.*

Proof. i) By definition.

ii) \Leftarrow) As Proposition 2.1.6. \Rightarrow) Let $g \in C_c^\infty(\mathbb{R})_e^\delta$ such that $\text{supp} g \subset \bar{B}(0, R)$. Then $\mathcal{F}g \in \mathcal{H}(\mathbb{C})_e^\delta$, with matrix entries $\{\mathcal{F}g\}_{i,j}$ in $\mathcal{H}^R(\mathbb{C})$. Now consider the function $G(z, kM) = \text{Tr}(\delta(k)\{\mathcal{F}g\}(z))$ in $\mathcal{H}_e^R(\mathbb{C} \times B)$. The proof of Theorem 2.3.6 yields that:

$$\mathcal{T}^{-1}g(x) = \mathcal{H}^{-1}G(x),$$

and then by Proposition 2.1.7, we see that $\text{supp} \mathcal{T}^{-1}g \subset \bar{B}(0, R)$. \square

Now we arrive at the essential theorem:

Theorem 2.3.9. *Let $0 < p \leq 2, \varepsilon = \frac{2}{p} - 1$. Then the δ -spherical transform \mathcal{H}^δ is a topological isomorphism between $\mathcal{S}^p(G/K)_\delta$ and $\mathcal{S}(i\mathbb{R}_\varepsilon)_e^\delta$. The inverse transform is given by Theorem 2.3.6.*

Proof. a) We can write \mathcal{H}^δ as a composition of continuous operators:

$$\mathcal{H}^\delta = ev \circ P^\delta \mathcal{H}.$$

Hence \mathcal{H}^δ is a injective continuous homomorphism into $\mathcal{S}(i\mathbb{R}_\varepsilon)_e^\delta$.

b) We now want to show surjectivity. By density, Theorem 2.2.10i), it is enough to show that the inverse δ -spherical transform is continuous as a map from $\mathcal{H}(\mathbb{C})_e^\delta$ to $C_c^\infty(G/K)_\delta$, with the topologies induced by $\mathcal{S}(i\mathbb{R}_\varepsilon)_e^\delta$ and $\mathcal{S}^p(G/K)_\delta$. So let $f \in C_c^\infty(G/K)_\delta$, $h = \mathcal{H}^\delta f$ and $H = \mathcal{T}f$ as in the commuting diagram below.

$$\begin{array}{ccc} & h \in \mathcal{H}(\mathbb{C})_e^\delta & \\ \mathcal{H}^\delta \nearrow \cong & & \nwarrow \cong \mathcal{F} \\ f \in C_c^\infty(G/K)_\delta & \xrightarrow[\mathcal{T}]{\cong} & H \in C_c^\infty(\mathbb{R})_e^\delta \end{array}$$

Let $\sigma_{D,E,N}^p$ be a seminorm for $\mathcal{S}^p(G/K)$:

$$\sigma_{D,E,N}^p(f) = \sup_{g \in G} (1 + |g|)^N \varphi_o(g)^{-\frac{2}{p}} |f(D; g; E)|.$$

We will consider the function: $F(g) = (1 + |g|)^N \varphi_o(g)^{-\frac{2}{p}} |f(D; g; E)|$. Our goal now is to estimate F on the intervals $|g| \in]j, j+1]$. Remark that all positive constants appearing below may depend on N and p , but not on f .

Step 1 ($[0, 2]$): By the inversion formula, Theorem 2.3.6, we get:

$$f(D; g; E) = cTr \left\{ \int_0^\infty \Phi_{i\nu, \delta}(D; g; E) h(i\nu) |c(i\nu)|^{-2} d\nu \right\}.$$

Using Lemma 2.1.9 iii) and Proposition 2.2.11, on the functions $\phi = \delta_{l,1}$, we get:

$$|f(D; g; E)| \leq c\varphi_o(g) \int_{\mathbb{R}_+} (1 + |\nu|)^R \sum_{l=1}^{d(\delta)} |h_{1,l}(i\nu)| d\nu,$$

for some $R \in \mathbb{N}$. We thus get:

$$\begin{aligned} \sup_{|g| \in [0, 2]} F(x) &\leq c \int_{\mathbb{R}_+} (1 + |\nu|)^R \sum_{l=1}^{d(\delta)} |h_{1,l}(i\nu)| d\nu \\ &\leq \sup_{\nu \in \mathbb{R}_+} (1 + |\nu|)^{R+2} \sum_{l=1}^{d(\delta)} |h_{1,l}(i\nu)| \\ &= c \sum_{l=1}^{d(\delta)} \tau_{1, R+2}^o(h_{1,l}), \end{aligned}$$

using the compactness of $[0, 2]$.

Step 2: For $j \in \mathbb{N}$ we introduce an even auxiliary function $\omega_j \in C_c^\infty(\mathbb{R})$ defined such that:

$$\omega_j(t) = \begin{cases} 1, & t \in [0, j-1[\\ 0, & t \in [j, \infty[\end{cases},$$

and $\omega_{j+1}(t) = \omega_j(t-1)$. We write H as:

$$\begin{aligned} H(t) &= p_\delta \left(-\frac{\partial}{\partial t} \right) \{ (1 - \omega_j)(t) \mathcal{F}^{-1} \{ h(\cdot) p_\delta^{-1}(-\cdot) \} (t) \} \\ &\quad + p_\delta \left(-\frac{\partial}{\partial t} \right) \{ \omega_j(t) \mathcal{F}^{-1} \{ h(\cdot) p_\delta^{-1}(-\cdot) \} (t) \}. \end{aligned}$$

Consider the functions:

$$\begin{aligned} H^j(t) &= p_\delta \left(-\frac{\partial}{\partial t} \right) \{ (1 - \omega_j)(t) \mathcal{F}^{-1} \{ h(\cdot) p_\delta^{-1}(-\cdot) \} (t) \}. \\ h^j(\nu) &= \mathcal{F} H^j(\nu) = p_\delta(-\nu) \{ \mathcal{F} \{ (1 - \omega_j)(\cdot) \mathcal{F}^{-1} \{ h(\cdot) p_\delta^{-1}(-\cdot) \} \} \}(\nu). \end{aligned}$$

We observe that $\{ (1 - \omega_j)(t) \mathcal{F}^{-1} \{ h(\cdot) p_\delta^{-1}(-\cdot) \} (t) \}$ is an even function. For h^j to be in $\mathcal{H}(\mathbb{C})_e^\delta$, it has to satisfy: $p_\delta(-\nu) h^j(-\nu) = p_\delta(\nu) h^j(\nu)$.

$$\begin{aligned} p_\delta(-\nu) h^j(-\nu) &= p_\delta(-\nu) p_\delta(\nu) \{ \mathcal{F} \{ (1 - \omega_j)(\cdot) \mathcal{F}^{-1} \{ h(\cdot) p_\delta^{-1}(-\cdot) \} \} \}(-\nu) \\ &= p_\delta(\nu) p_\delta(-\nu) \{ \mathcal{F} \{ (1 - \omega_j)(\cdot) \mathcal{F}^{-1} \{ h(\cdot) p_\delta^{-1}(-\cdot) \} \} \}(\nu) \\ &= p_\delta(\nu) h^j(\nu), \end{aligned}$$

and hence $h^j \in \mathcal{H}(\mathbb{C})_e^\delta$ and $H^j \in C_c^\infty(\mathbb{R})_e^\delta$. Let f^j be the corresponding element of $C_c^\infty(G/K)_\delta$. Since ω_j has support in $[0, j]$, Proposition 2.3.8 tells us that f may differ from f^j only inside $K \times [0, j] \times K$.

Step 3 ($[j, j+1]$): As in step 1 we get:

$$\begin{aligned} |f^j(D; g; E)| &\leq c\varphi_o(g) \int_{\mathbb{R}_+} (1+|\nu|)^R \sum_{l=1}^{d(\delta)} |h_{1,l}^j(i\nu)| d\nu \\ &\leq c\varphi_o(g) \sum_{l=1}^{d(\delta)} \tau_{1,R+2}^o(h_{1,l}^j). \end{aligned}$$

It follows that:

$$\sup_{|g| \in [j, j+1]} |F(g)| \leq c j^N e^{\varepsilon \rho j} \sum_{l=1}^{d(\delta)} \tau_{1,R+2}^o(h_{1,l}^j),$$

by estimates on φ_ν (Lemma 2.1.9).

Step 4: We now want to find a constant c and $m, n \in \mathbb{N}$ s.t.

$$j^N e^{\varepsilon \rho j} \tau_{1,R+2}^o(h_{1,l}^j) \leq c \sum_{k=0}^m \sup_{\nu \in i\mathbb{R}_\varepsilon} (1+|\nu|)^n |\nabla^k h_{1,l}(\nu)|,$$

for all l . In the following ∇ will denote either $\nabla = \frac{d}{d\nu}$ or $\nabla = \frac{d}{dt}$. The connection between $h_{1,l}^j$ and $H_{1,l}^j$ is:

$$h_{1,l}^j(\nu) = \mathcal{F}H_{1,l}^j(\nu) = \int_{\mathbb{R}} H_{1,l}^j(t) e^{-\nu t} dt,$$

and

$$H_{1,l}^j(t) = \mathcal{F}^{-1}h_{1,l}^j(t) = \frac{1}{2\pi} \int_{\mathbb{R}} h_{1,l}^j(i\nu) e^{i\nu t} d\nu.$$

Thus:

$$\begin{aligned} \tau_{1,R+2}^o(h_{1,l}^j) &= \sup_{\nu \in i\mathbb{R}} (1+|\nu|)^{R+2} |h_{1,l}^j(\nu)| \\ &= \sup_{\nu \in i\mathbb{R}} (1+|\nu|)^{R+2} |\mathcal{F}H_{1,l}^j(\nu)| \\ &= \sup_{\nu \in i\mathbb{R}} (1+|\nu|)^{R+2} \left| \int_{\mathbb{R}} H_{1,l}^j(t) e^{-\nu t} dt \right| \\ &\leq c \sum_{k=0}^{R+2} \int_{\mathbb{R}} |\nabla^k H_{1,l}^j(t)| dt \\ &\leq c \sup_{t \geq 0} \sum_{k=0}^{R+2} (1+t)^2 |\nabla^k H_{1,l}^j(t)|. \end{aligned}$$

We compute the derivatives of $H_{1,l}^j(\cdot)$ by the Leibniz rule. $1 - \omega_j$ and its derivatives vanish on $[0, j-1]$, and are bounded on $[j-1, \infty[$, uniformly in j . Consequently:

$$\begin{aligned} &j^N e^{\varepsilon \rho t} \tau_{1,R+2}^o(h_{1,l}^j) \\ &\leq c \sum_{k=0}^{R+2} \sup_{t \geq j-1} (1+t)^{N+2} e^{\varepsilon \rho t} \left| \nabla^k p_\delta \left(-\frac{\partial}{\partial t} \right) \mathcal{F}^{-1} \{h_{1,l}(\cdot) p_\delta^{-1}(-\cdot)\}(t) \right| \\ &\leq c \sum_{k=0}^{R+2} \sup_{t \geq 0} (1+t)^{N+2} e^{\varepsilon \rho t} \left| \nabla^k p_\delta \left(-\frac{\partial}{\partial t} \right) \mathcal{F}^{-1} \{h_{1,l}(\cdot) p_\delta^{-1}(-\cdot)\}(t) \right|. \end{aligned}$$

Let P and Q be polynomials, then by Cauchy's Theorem, shifting integral from $i\nu$ to $i\nu + \rho\varepsilon$, we get:

$$\begin{aligned} & P(t)e^{\rho\varepsilon t}Q\left(\frac{\partial}{\partial t}\right)\mathcal{F}^{-1}\{h_{1,l}(\cdot)p_\delta^{-1}(-\cdot)\}(t) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}P\left(i\frac{\partial}{\partial\nu}\right)\{Q(i\nu - \rho\varepsilon)h_{1,l}(i\nu - \rho\varepsilon)p_\delta^{-1}(-i\nu + \rho\varepsilon)\}e^{i\nu t}d\nu. \end{aligned}$$

Hence we get:

$$\begin{aligned} & \sum_{k=0}^{R+2}\sup_{t\geq 0}(1+t)^{N+2}e^{\varepsilon\rho t}\left|\nabla^k p_\delta\left(-\frac{\partial}{\partial t}\right)\mathcal{F}^{-1}\{h_{1,l}(\cdot)p_\delta^{-1}(-\cdot)\}(t)\right| \\ & \leq c\sum_{k=0}^{N+2}\int_{\mathbb{R}}(1+|\nu|)^{\tilde{R}+2}|\{\nabla^k(h_{1,l}p_\delta^{-1}(-\cdot))\}(i\nu - \rho\varepsilon)|d\nu, \end{aligned}$$

where $\tilde{R} = \deg p_\delta + R$. Now the remaining problem is to estimate $h_{1,l}(\nu)p_\delta^{-1}(-\nu)$ and derivatives on the boundary of the tube $i\mathbb{R}_\varepsilon$. Recall that the polynomials $p_\delta(\nu)$ are of the form:

$$p_\delta(\nu) = (\nu + \rho + s - 1) \cdots (\nu + \rho), \quad s \in \mathbb{N} \quad \text{or} \quad p_\delta(\nu) \equiv 1.$$

So we see that $p_\delta^{-1}(-\cdot)$ has at most one singularity on the boundary. Assume that we a singularity ν_o on the boundary of $i\mathbb{R}_\varepsilon$. Since $h_{1,l}(\nu_o) = 0$, we can write $h_{1,l}$ as:

$$\begin{aligned} h_{1,l}(\nu) &= (\nu - \nu_o)h_{1,l}^\#(\nu) \text{ and} \\ h_{1,l}^\#(\nu) &= \int_0^1 h'_{1,l}(\nu_o + t(\nu - \nu_o))dt. \end{aligned}$$

We thus have: $h_{1,l}(\nu)p_\delta^{-1}(-\nu) = p_\delta^{-1}(-\nu)(\nu - \nu_o)h_{1,l}^\#(\nu)$, where the function $p_\delta^{-1}(-\nu)$ ($\nu - \nu_o$) has no singularity on the boundary. Furthermore we see that the N first derivatives of $h_{1,l}^\#$ can be estimated by the $(N+1)$ first derivatives of $h_{1,l}$. Hence we get:

$$\begin{aligned} & \sum_{k=0}^{N+2}\int_{\mathbb{R}}(1+|\nu|)^{\tilde{R}+2}|\{\nabla^k(h_{1,l}p_\delta^{-1}(-\cdot))\}(i\nu - \rho\varepsilon)|d\nu \\ & \leq c\sum_{k=0}^{N+3}\int_{\mathbb{R}}(1+|\nu|)^{\tilde{R}+2}|\{\nabla^k h_{1,l}\}(i\nu - \rho\varepsilon)|d\nu \\ & \leq c\sum_{k=0}^{N+3}\sup_{\nu \in i\mathbb{R}_\varepsilon}(1+|\nu|)^{\tilde{R}+4}|\nabla^k h_{1,l}(\nu)|. \end{aligned}$$

All in all this gives, for some positive constant c

$$\sigma_{D,E,N}^p(f) \leq c\sum_{l=1}^{d(\delta)}\sum_{k=0}^{N+3}\sup_{\nu \in i\mathbb{R}_\varepsilon}(1+|\nu|)^{\tilde{R}+4}|\nabla^k h_{1,l}(\nu)|.$$

This concludes the proof. \square

Corollary 2.3.10. *Let $0 < p \leq 2$. The Fourier transform \mathcal{H} is a topological isomorphism between $\mathcal{S}^p(G/K)_\delta$ and $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_{\delta,e}$, taking $C_c^\infty(G/K)_\delta$ to $\mathcal{H}(\mathbb{C} \times B)_{\delta,e}$. The inverse transform is given by Theorem 2.1.2.*

Proof. Theorem 2.3.9 and the isomorphisms discussed in Proposition 2.2.6 and thereafter. Actually let $h \in \mathcal{H}(\mathbb{C} \times B)_{\delta,e}$, then we get the following estimate on $f = \mathcal{H}^{-1}h \in C_c^\infty(G/K)_\delta$, for some positive constant c

$$\sigma_{D,E,N}^p(f) \leq cd(\delta)^2 \sum_{l=1}^{d(\delta)} \sum_{k=0}^{N+3} \sup_{\nu \in i\mathbb{R}_\varepsilon, k \in K} (1 + |\nu|)^{\tilde{R}+4} |\nabla^k h_{1,l}(\nu, kM)|.$$

□

Let $\mathcal{S}^p(G/K)_K$ and $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_{e,K}$ denote the K -finite elements of $\mathcal{S}^p(G/K)$ and $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_e$, then we finally get the K -finite version of Theorem 1:

Theorem 2.3.11. *Let $0 < p \leq 2$. The Fourier transform \mathcal{H} is a topological isomorphism between $\mathcal{S}^p(G/K)_K$ and $\mathcal{S}(i\mathbb{R}_\varepsilon \times B)_{e,K}$. The inverse transform is given by Theorem 2.1.2.*

Remark: It is not possible via a density argument to use the above theorem to prove Theorem 1 in general, since we don't have a polynomial or uniform bound in the estimate above for the various K -types. In estimating the derivatives of $H_{1,l}^j$, we used boundedness of finite derivations of ω_j , but unfortunately we cannot have a uniform or polynomial bound on the derivatives of ω_j , as this would imply analyticity of ω_j . Hence, as there is no bound on the degree of the polynomials p_δ , we cannot have a uniform or polynomial bound on the constants c .

3. General rank.

We will in this Section sketch how to remove the $\dim \mathfrak{a} = 1$ condition in Section 2. So let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of \mathfrak{g} , and fix a maximal subspace \mathfrak{a} in \mathfrak{p} ($\dim \mathfrak{a} \geq 1$). Denote its real dual by \mathfrak{a}^* and its complex dual by $\mathfrak{a}_\mathbb{C}^*$. Let $\Sigma \subset \mathfrak{a}^*$ be the root system of $(\mathfrak{g}, \mathfrak{a})$, and let W be the Weyl group associated to Σ ($\cong N_K(\mathfrak{a})/Z_K(\mathfrak{a})$). Choose a set Σ^+ of positive roots, and let $\mathfrak{a}_+ \subset \mathfrak{a}$ and $\mathfrak{a}_+^* \subset \mathfrak{a}^*$ be the corresponding positive Weyl chambers. We will define ρ as in Section 2. Fix $\varepsilon \geq 0$, let $C^{\varepsilon\rho}$ be the convex hull of the set $W \cdot \varepsilon\rho$ in \mathfrak{a}^* and let $\mathfrak{a}_\varepsilon^* = \mathfrak{a}^* + iC^{\varepsilon\rho}$ be the tube in $\mathfrak{a}_\mathbb{C}^*$ with basis $C^{\varepsilon\rho}$.

In the following, the variable $\nu \in \mathbb{C}$ in Section 2 will correspond to the variable $i\lambda \in \mathfrak{a}_\mathbb{C}^*$ and differentiation with $P(\frac{\partial}{\partial \nu})$ will correspond to $P(\frac{\partial}{\partial \lambda})$, where $P \in \mathcal{S}(\mathfrak{a}^*)$. On nice functions on $X = G/K$, we will define the Fourier transform as:

$$\mathcal{H}f(\lambda, b) = \hat{f}(\lambda, b) = \int_X f(x) e^{(-i\lambda + \rho)(A(x, b))} dx,$$

for $\lambda \in \mathfrak{a}_\mathbb{C}^*$, $b \in B$, when welldefined, see [He2, Chapter III, §1]. Inversion formulas, Plancherel formulas and a Paley-Wiener Theorem can be found in [He2, Chapter.III]. For $\varepsilon \geq 0$, let $\mathcal{S}(\mathfrak{a}_\varepsilon^* \times B)$ be the Schwartz space on $\mathfrak{a}_\varepsilon^* \times B$ (replace \mathbb{R}_ε with $\mathfrak{a}_\varepsilon^*$ in definition 2.2.3), and let $\mathcal{S}(\mathfrak{a}_\varepsilon^* \times B)^W$ be the subspace of $\mathcal{S}(\mathfrak{a}_\varepsilon^* \times B)$ of functions satisfying the symmetry condition (SC):

$$\int_B e^{(si\lambda + \rho)(A(x, b))} \psi(s\lambda, b) db = \int_B e^{(i\lambda + \rho)(A(x, b))} \psi(\lambda, b) db,$$

for $s \in W$, $\lambda \in \mathfrak{a}_\varepsilon^*$ and $x \in X$. For $0 < p \leq 2$, $\mathcal{S}^p(G/K)$ denotes the L^p -Schwartz space on G/K . The group K acts naturally on $\mathcal{S}^p(G/K)$ and $\mathcal{S}(\mathfrak{a}_\varepsilon^* \times B)^W$ (on the second variable, the action preserving the symmetry condition (SC)). Let $\mathcal{S}^p(G/K)_K$ and $\mathcal{S}(\mathfrak{a}_\varepsilon^* \times B)_K^W$ denote the K -finite elements of $\mathcal{S}^p(G/K)$ and $\mathcal{S}(\mathfrak{a}_\varepsilon^* \times B)^W$ respectively. In this setup, Theorem 2.3.2 and Theorem 2.3.11 become:

Theorem 3.1. *Let $0 < p \leq 2$ and $\varepsilon = \frac{2}{p} - 1$.*

- (i) *The Fourier transform is an injective and continuous homomorphism from $\mathcal{S}^p(G/K)$ into $\mathcal{S}(\mathfrak{a}_\varepsilon^* \times B)^W$.*
- (ii) *The Fourier transform \mathcal{H} is a topological isomorphism between $\mathcal{S}^p(G/K)_K$ and $\mathcal{S}(\mathfrak{a}_\varepsilon^* \times B)_K^W$.*
- (iii) *The inverse transform is given by: $(\psi \in \mathcal{S}(\mathfrak{a}_\varepsilon^* \times B)_K^W)$*

$$\mathcal{H}^{-1}\psi(x) = c \int_{\mathfrak{a}^* \times B} e^{(i\lambda + \rho)(A(x,b))} \hat{f}(\lambda, b) |c(\lambda)|^{-2} d\lambda db.$$

As in Section 1, the difficult part is to prove (ii) and (iii). Let $C_c^\infty(X)$ be as usual. Consider the Paley-Wiener space $\mathcal{H}(\mathfrak{a}_\mathbb{C}^* \times B)$ (replace \mathbb{C} with $\mathfrak{a}_\mathbb{C}^*$ and $|Re \cdot|$ with $|Im \cdot|$), and denote by $\mathcal{H}(\mathfrak{a}_\mathbb{C}^* \times B)^W$ the subspace of $\mathcal{H}(\mathfrak{a}_\mathbb{C}^* \times B)$ of functions satisfying the symmetry condition (SC) for $\lambda \in \mathfrak{a}_\mathbb{C}^*$. Then we have:

Theorem 3.2. *Let $0 < p \leq 2$ and $\varepsilon \geq 0$. Then:*

- (i) *$C_c^\infty(X)$ is a dense subspace of $\mathcal{S}^p(G/K)$.*
- (ii) *$\mathcal{H}(\mathfrak{a}_\mathbb{C}^* \times B)$ is a dense subspace of $\mathcal{S}(\mathfrak{a}_\varepsilon^* \times B)$.*
- (iii) *$\mathcal{H}(\mathfrak{a}_\mathbb{C}^* \times B)^W$ is a dense subspace of $\mathcal{S}(\mathfrak{a}_\varepsilon^* \times B)^W$.*

Proof. (i) See Lemma 2.2.2.

(ii) See Lemma 2.2.4 and [An].

To prove (iii), we will consider the symmetry conditions (SC) for various K -types, and again they reduce to polynomial symmetry conditions. Recall the definitions of the various projections in Section 2, §2, and define the subspaces: $\mathcal{S}(\mathfrak{a}_\varepsilon^* \times B)_\delta^W$, $\mathcal{S}(\mathfrak{a}_\varepsilon^* \times B)^{\delta, W}$, $\mathcal{H}(\mathfrak{a}_\mathbb{C}^* \times B)_\delta^W$ and $\mathcal{H}(\mathfrak{a}_\mathbb{C}^* \times B)^{\delta, W}$ of functions of K -type δ satisfying (SC). Using the evaluation map, we get isomorphisms between $\mathcal{S}(\mathfrak{a}_\varepsilon^* \times B)^{\delta, W}$ and $\mathcal{S}(\mathfrak{a}_\varepsilon^*)_W^\delta$, and between $\mathcal{H}(\mathfrak{a}_\mathbb{C}^* \times B)^{\delta, W}$ and $\mathcal{H}(\mathfrak{a}_\mathbb{C}^*)_W^\delta$, where:

$$\begin{aligned} \mathcal{H}(\mathfrak{a}_\mathbb{C}^*)_W^\delta &\equiv \{F \in \mathcal{H}(\mathfrak{a}_\mathbb{C}^*, \text{Hom}(V_\delta, V_\delta^M)) : (Q^\delta)^{-1}F \text{ is } W\text{-invariant}\} \\ \mathcal{S}(\mathfrak{a}_\varepsilon^*)_W^\delta &\equiv \{F \in \mathcal{S}(\mathfrak{a}_\varepsilon^*, \text{Hom}(V_\delta, V_\delta^M)) : (Q^\delta)^{-1}F \text{ is } W\text{-invariant}\}. \end{aligned}$$

Here $Q^\delta(\lambda)$ is the Kostant Q -polynomial, see [He2, Chapter III, §2,3,5]. Fix an orthonormal basis $v_1, \dots, v_{d(\delta)}$ of V_δ such that $v_1, \dots, v_{l(\delta)}$ span V_δ^M . Then the members of $\mathcal{H}(\mathfrak{a}_\mathbb{C}^*)_W^\delta$ resp. $\mathcal{S}(\mathfrak{a}_\varepsilon^*)_W^\delta$ become matrix valued holomorphic functions on $\mathfrak{a}_\mathbb{C}^*$, respectively on $\mathfrak{a}_\varepsilon^*$, and Q^δ is an $l(\delta) \times l(\delta)$ matrix whose entries are polynomials on $\mathfrak{a}_\mathbb{C}^*$ ([He2, p.287]). Let $\mathcal{H}(\mathfrak{a}_\mathbb{C}^*)_e^\delta$ and $\mathcal{S}(\mathfrak{a}_\varepsilon^*)_e^\delta$ denote the spaces of Weyl group invariants in $\mathcal{H}(\mathfrak{a}_\mathbb{C}^*, \text{Hom}(V_\delta, V_\delta^M))$, respectively in $\mathcal{S}(\mathfrak{a}_\varepsilon^*, \text{Hom}(V_\delta, V_\delta^M))$ (corresponding to $Q^\delta \equiv I$), then we get the crucial fact:

Fact: The mapping $\psi(\lambda) \mapsto Q^\delta(\lambda)\psi(\lambda)$ is a homeomorphism of $\mathcal{S}(\mathfrak{a}_\varepsilon^*)_e^\delta$ onto $\mathcal{S}(\mathfrak{a}_\varepsilon^*)_W^\delta$, taking $\mathcal{H}(\mathfrak{a}_\mathbb{C}^*)_e^\delta$ onto $\mathcal{H}(\mathfrak{a}_\mathbb{C}^*)_W^\delta$.

The map clearly is into, and from [He2, Chapter III, Lemma 5.12] we see that the map takes $\mathcal{H}(\mathfrak{a}_\mathbb{C}^*)_e^\delta$ onto $\mathcal{H}(\mathfrak{a}_\mathbb{C}^*)_W^\delta$. We can write:

$$Q^\delta(\lambda)^{-1} = Q_c(\lambda)(\det Q^\delta(\lambda))^{-1},$$

where Q_c is a matrix whose entries are polynomials on $\mathfrak{a}_\mathbb{C}^*$, and $\det Q^\delta(\lambda)$ is a product of polynomials coming from the rank one reduction, see [He2, p.263 (50)] and [He2, Chapter III, Theorem 4.2]. From [He2, §11], we conclude that $\det Q^\delta(\lambda)$ is non-zero in a neighbourhood of $\mathfrak{a}^* + i\overline{\mathfrak{a}_+^*}$, and considering only one Weyl chamber, using Weyl group invariance, we conclude the result. Using the matrix

valued classical Fourier transform (see below) we conclude that $\mathcal{H}(\mathfrak{a}_{\mathbb{C}}^*)_{\varepsilon}^{\delta}$ is dense in $\mathcal{S}(\mathfrak{a}_{\varepsilon}^*)_{\varepsilon}^{\delta}$, and hence $\mathcal{H}(\mathfrak{a}_{\mathbb{C}}^*)_{\varepsilon}^{\delta}$ is dense in $\mathcal{S}(\mathfrak{a}_{\varepsilon}^*)_{\varepsilon}^{\delta}$. Elaborating on these results, as in Section 2, we get iii). \square

Let \mathcal{R} be the Radon transform (replace $a_t \in A$ with $\exp(H)$, $H \in \mathfrak{a}$). Let $\phi \in C_c^{\infty}(\mathfrak{a} \times B)$, then the "classical" Fourier transform on $\mathfrak{a} \times B$ is defined as:

$$\mathcal{F}\phi(\lambda, b) = \int_{\mathfrak{a}} \phi(H, b) e^{-i\lambda(H)} dH, \lambda \in \mathfrak{a}_{\mathbb{C}}^*, b \in B.$$

We have the following commutative diagram:

$$\begin{array}{ccc} & \mathcal{H}(\mathfrak{a}_{\mathbb{C}}^* \times B)^W & \\ \mathcal{H} \nearrow & & \nwarrow \mathcal{F} \\ C_c^{\infty}(G/K) & \xrightarrow{\mathcal{R}} & \mathcal{R}C_c^{\infty}(G/K) \subset C_c^{\infty}(\mathfrak{a} \times B) \end{array}$$

As in Section 2, all transforms preserves K -types. We define the δ -spherical transform:

$$\mathcal{H}^{\delta}f(\lambda) = d(\delta) \int_X f(x) \Phi_{\lambda, \delta}(x)^* dx$$

where $\Phi_{\lambda, \delta}$ is the generalized sperical function:

$$\Phi_{\lambda, \delta} = \int_K e^{(i\lambda + \rho)(A(x, kM))} \delta(k) dk, x \in X,$$

see [He2, Chapter 3, §2, §5].

Theorem 3.3. *The δ -spherical transform $f \mapsto \mathcal{H}^{\delta}f$ is a bijection of $C_c^{\infty}(G/K)_{\delta}$ onto $\mathcal{H}(\mathfrak{a}_{\mathbb{C}}^*)_{\varepsilon}^{\delta}$.*

Proof. [He2, Chapter III, Theorem 5.11]. \square

Defining an operator \mathcal{T} as in Section 2, and considering the "classical" Fourier transform on matrix coefficients

$$\mathcal{F}\phi(\lambda, b) = \int_{\mathfrak{a}} \phi(H) e^{i\lambda(H)} dH, \lambda \in \mathfrak{a}_{\mathbb{C}},$$

we get the commuting diagram:

$$\begin{array}{ccc} & \psi \in \mathcal{H}(\mathfrak{a}_{\mathbb{C}}^*)_{\varepsilon}^{\delta} & \\ \mathcal{H}^{\delta} \nearrow \cong & & \nwarrow \mathcal{F} \\ f \in C_c^{\infty}(G/K)_{\delta} & \xrightarrow[\mathcal{T}]{\cong} & \phi \in C_c^{\infty}(\mathfrak{a})_{\varepsilon}^{\delta} \end{array} \quad (4)$$

The subscript W on $C_c^{\infty}(\mathfrak{a})_{\varepsilon}^{\delta}$ indicates some kind of symmetry condition. To show Theorem 3.1 ii)+iii), we are left to show that $(\mathcal{H}^{\delta})^{-1}$ is a continuous homomorphism from $\mathcal{H}(\mathfrak{a}_{\mathbb{C}}^*)_{\varepsilon}^{\delta}$ into $C_c^{\infty}(G/K)_{\delta}$ in the induced topologies. Again we want to use a cut-off function, as in Section 2 and [An]. To do so, we need to reformulate the Paley-Wiener Theorems for the various transforms. Define convex W -invariant sets in \mathfrak{a} and G by: $\mathfrak{a}_R = \{H \in \mathfrak{a} | \rho(w \cdot H) \leq R, \forall w \in W\}$, and $G_R = K(\exp \mathfrak{a}_R)K$. Furthermore define the gauge associated to \mathfrak{a}_R by $q(\lambda) = \sup_{H \in \mathfrak{a}_R} \lambda(H)$.

Theorem 3.4. *The "classical" Fourier transform \mathcal{F} is an isomorphism between the space of all functions $\phi \in C_c^\infty(\mathfrak{a})$, such that $\text{supp } \phi \subset \mathfrak{a}_R$, and the space of all entire functions ψ on $\mathfrak{a}_\mathbb{C}^*$, such that*

$$\sup_{\lambda \in \mathfrak{a}_\mathbb{C}^*} e^{-q(\text{Im } \lambda)} (1 + \|\lambda\|)^N |\psi(\lambda)| < \infty, \quad (5)$$

for all $N \in \mathbb{N}$.

Theorem 3.5. *The δ -spherical transform \mathcal{H}^δ is an isomorphism between the space of functions $f \in C_c^\infty(G/K)$, with $\text{supp } f \subset G_R$, and the space of functions $\psi \in \mathcal{H}(\mathfrak{a}_\mathbb{C}^*)^\delta_W$, with matrix entries satisfying (5).*

Proof. A slight modification of the proof of [He2, Chapter III, Theorem 5.11], using the ideas from [An, p. 341-2]. \square

Corollary 3.6. *The transform \mathcal{T} is an isomorphism between the spaces $C_c^\infty(G/K)$ and $C_c^\infty(\mathfrak{a})^\delta_W$. Moreover $\text{supp } f \subset G_R$ if and only if $\text{supp } \mathcal{T}f \subset \mathfrak{a}_R$.*

Consider the commuting diagram (4). Given a seminorm σ in $\mathcal{S}^p(G/K)$, we shall find a seminorm τ on $\mathcal{S}(\mathfrak{a}_\mathbb{C}^*)^\delta_W$, such that $\sigma(f) \leq \tau(\psi)$, for all f and ψ . Instead of looking at the intervals $[0, j]$ and $[j, j+1]$, as we did in Section 2, we will consider the sets G_j and $G_{j+1} \setminus G_j$. The crucial point in the proof is then to estimate $f(D; g; E)$ on $G_{j+1} \setminus G_j$. Let $\omega \in C^\infty(\mathbb{R})$, with $\omega \equiv 0$ on $]-\infty; 0]$, and $\omega \equiv 1$ on $[1; \infty[$. Introduce a W -invariant function in $C_c^\infty(\mathfrak{a})$ by:

$$\omega_j(H) = \prod_{w \in W} \omega(j - \rho(w \cdot H)).$$

We see that $\omega_j \equiv 1$ on \mathfrak{a}_{j-1} , and $\omega_j \equiv 0$ outside \mathfrak{a}_j . Moreover ω_j and all its derivatives are bounded in j . We decompose ϕ as:

$$\begin{aligned} \phi(H) &= Q^\delta \left(-i \frac{\partial}{\partial H} \right) \{ (1 - \omega_j)(H) \mathcal{F}^{-1} \{ (Q^\delta)^{-1}(\cdot) \psi(\cdot) \} (H) \} \\ &\quad + Q^\delta \left(-i \frac{\partial}{\partial H} \right) \{ \omega_j(H) \mathcal{F}^{-1} \{ (Q^\delta)^{-1}(\cdot) \psi(\cdot) \} (H) \}. \end{aligned}$$

Consider the functions

$$\begin{aligned} \phi_j(H) &= Q^\delta \left(-i \frac{\partial}{\partial H} \right) \{ (1 - \omega_j)(H) \mathcal{F}^{-1} \{ (Q^\delta)^{-1}(\cdot) \psi(\cdot) \} (H) \}, \\ \psi_j(\lambda) &= \mathcal{F} \phi_j(\lambda) = Q^\delta(\lambda) \{ \mathcal{F} \{ (1 - \omega_j)(\cdot) \mathcal{F}^{-1} \{ (Q^\delta)^{-1}(\cdot) \psi(\cdot) \} \} \} (\lambda), \\ f_j(x) &= ((\mathcal{H}^\delta)^{-1} \psi_j)(x) = (\mathcal{T}^{-1} \phi_j)(x). \end{aligned}$$

We see that $\psi_j \in \mathcal{H}(\mathfrak{a}_\mathbb{C}^*)^\delta_W$ and $\phi_j \in C_c^\infty(\mathfrak{a})^\delta_W$. Since ω_j has support in \mathfrak{a}_j , Corollary 3.6 tells us that f may differ from f_j only inside G_j . Continuing as in [An] or Section 2, using elementary Fourier analysis, we see that the remaining problem is to estimate $(Q^\delta)^{-1}(\cdot) \psi(\cdot)$ and its derivatives on the boundary of $\mathfrak{a}_\mathbb{C}^*$, with similar estimates on $\psi(\cdot)$. Following the argument of Theorem 2.3.9, using the knowledge from the proof of Theorem 3.2, we reach the result.

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